Maths IB$_2$AA HL

Test 4

Thursday 18.11.2025

Subjects : *Differential equations*

Total : / 17

ANSWERS

Problem 1 IB Nov21

[max / 8 marks]

Solve the differential equation $\dfrac{dy}{dx} = \dfrac{\ln(2x)}{x^2} - \dfrac{2y}{x}$, $x > 0$, given that $y = 4$ at $x = \frac{1}{2}$.

$\boxed{\textit{Integrant factor } I(x) = e^{\int -P(x)dx}}$ Here $I(x) = e^{2\ln(x)+c} = Cx^2$

Then the ED becomes : $x^2 \dfrac{dy}{dx} = x^2\left(\dfrac{\ln(2x)}{x^2} - \dfrac{2y}{x}\right)$

$$x^2\left(\frac{dy}{dx} + \frac{2y}{x}\right) = \ln(2x)$$

$$(x^2\, y)' = \ln(2x)$$

$$\Rightarrow x^2\, y = \int \ln(2x)\, dx + c = x(\ln(2x) - 1) + c$$

$$\Rightarrow y = \frac{x(\ln(2x) - 1) + c}{x^2} = \frac{(\ln(2x) - 1)}{x} + \frac{c}{x^2}$$

$$y = 4 \text{ at } x = \tfrac{1}{2} \Rightarrow 4 = \frac{\ln(1) - 1}{\frac{1}{2}} + \frac{c}{\left(\frac{1}{2}\right)^2} = -2 + 4c \Rightarrow c = \frac{6}{4}$$

$$\boxed{y = \frac{\ln(2x) - 1}{x} + \frac{3}{2x^2}}$$

Problem 2 IB May23

[max / 9 marks]

Consider the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2 + 3y^2}{xy}$, where $x > 0, y > 0$.

It is given that $y = 2$ when $x = 1$.

By solving the differential equation, show that $y = x\sqrt{\dfrac{9x^4 - 1}{2}}$. [8]

Find the value of y when $x = 1.1$. [1]

Let be $q = \frac{y}{x}$, then $\frac{dq}{dx} = \frac{y'x - y}{x^2} = \frac{y'}{x} - \frac{q}{x}$ then $y' = x\frac{dq}{dx} + q$

The DE becomes : $x\frac{dq}{dx} + q = \frac{1}{q} + 3q \quad \Rightarrow \quad x\frac{dq}{dx} = \frac{1}{q} + 2q \qquad \frac{1}{x}dx = \frac{1}{\frac{1}{q} + 2q}dq$

$$\Rightarrow \quad \int \frac{1}{x}dx = \int \frac{q}{1 + 2q^2}dq = \frac{1}{4}\int \frac{4q}{1 + 2q^2}dx$$

$$\Rightarrow \quad \ln(x) = \frac{1}{4}\ln(1 + 2q^2) + \mathrm{c}$$

$$\Rightarrow \quad x = C\sqrt[4]{1 + 2q^2} = C \cdot \sqrt[4]{1 + 2\left(\tfrac{y}{x}\right)^2}$$

$$\Rightarrow \quad x^4 = C^6\left(1 + 2\left(\tfrac{y}{x}\right)^2\right)$$

$$\Rightarrow \quad \left(1 + 2\left(\tfrac{y}{x}\right)^2\right) = \mathcal{C}\, x^4$$

$$\Rightarrow \quad 2\left(\tfrac{y}{x}\right)^2 = \mathcal{C}\, x^4 - 1 \qquad \Rightarrow \quad \boxed{y = \pm x\sqrt{\frac{\mathcal{C}x^4 - 1}{2}}}$$

At $y = 2$ when $x = 1 \Rightarrow 2 = \pm\sqrt{\frac{\mathcal{C} - 1}{2}} \Rightarrow \frac{\mathcal{C} - 1}{2} = 4 \qquad \mathcal{C} = 9 \qquad$ ok

When $x = \frac{11}{10}$, $\boxed{y = \frac{11}{1000}\sqrt{\frac{121769}{2}}}$