Maths IB_2 HL

Test 1

Subjects : *derivatives, implicit derivation* ...

Total : / 23

Tuesday 30.10.2025

ANSWERS

Problem 1

[/12 marks]

A curve C is defined implicitly by the equation $e^{\sin y} = \sin x + y^2$.

(a) Find the equation of the tangent to C at the point where $y = 0$ and $0 < x < \pi$.

(b) Find the equation of the normal to C at the point where $y = 0$ and $0 < x < \pi$.

$$e^{\sin(y)}\cos(y)y' = \cos(x) + 2yy'$$

$$\Rightarrow (e^{\sin(y)}\cos(y) - 2y)y' = \cos(x)$$

$$y'(x,y) = \frac{\cos(x)}{e^{\sin(y)}\cos(y) - 2y} \qquad \text{if } y = 0 \quad \text{then} \quad 1 = \sin(x) \quad \Rightarrow x = \frac{\pi}{2}$$

$$y'\left(\frac{\pi}{2}, 0\right) = \frac{\cos(\frac{\pi}{2})}{e^{\sin(0)}\cos(0) - 0} = \frac{0}{1} = 0$$

a) *Tangent*: (horizontal line)

then $y = 0x + b$ $\qquad 0 = +b \qquad b = -\frac{\pi}{2} \quad \Rightarrow \boxed{y = 0}$

b) *Normal*: (vertical line)

$\boxed{x = \frac{\pi}{2}}$

(c) Find the second derivative at the point P.

$$y'(x,y) = \frac{\cos(x)}{e^{\sin(y)}\cos(y) - 2y}$$

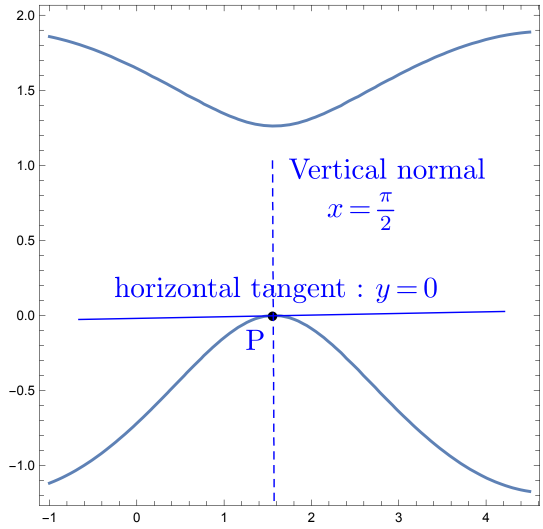
$$y''(x,y) = \frac{-\sin(x)(e^{\sin(y)}\cos(y) - 2y) - \cos(x)(e^{\sin(y)}\cos^2(y)y' - e^{\sin(y)}\sin(y)y' - 2y')}{(e^{\sin(y)}\cos(y) - 2y)^2}$$

$$= \frac{-\sin(x)(e^{\sin(y)}\cos(y) - 2y) - (\cos^3(x)e^{\sin(y)} + e^{\sin(y)}\sin(y) + 2)y'}{(e^{\sin(y)}\cos(y) - 2y)^2}$$

we replace y' ↙

$$= \frac{-\sin(x)(e^{\sin(y)}\cos(y) - 2y) - (\cos^3(x)e^{\sin(y)} + e^{\sin(y)}\sin(y) + 2)\frac{\cos(x)}{e^{\sin(y)}\cos(y) - 2y}}{(e^{\sin(y)}\cos(y) - 2y)^2}$$

d)



e) $y''(\frac{\pi}{2},0)=\dfrac{-\sin(\frac{\pi}{2})(e^{\sin(0)}\cos(0)-0)-\left(\cos^3(\frac{\pi}{2})e^{\sin(0)}+e^{\sin(y)}\sin(y)+2\right)\frac{\cos(\frac{\pi}{2})}{e^{\sin(0)}\cos(0)-0}}{(e^{\sin(0)}\cos(0)-0)^2}$

$=\dfrac{-1(1-0)-(0e^0+e^0\,0+2)\frac{0}{1-0}}{(1-0)^2}=\dfrac{-1}{1}$ $\boxed{=-1<0 \Rightarrow \text{concave up} \Rightarrow \text{We have a } \textit{maximum} \text{ at P}}$

Problem 2 [/4 marks]

Given that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{ky-x^2}{y^2-kx}$, $k>0$ when $x^3+y^3-6xy=0$, find the value of k.

$3x^2+3y^2\frac{dy}{dx}-6y-6x\frac{dy}{dx}=0 \quad \Rightarrow \frac{dy}{dx}(-6x+3y^2)=-3x^2+6y \quad \Rightarrow \frac{dy}{dx}=\frac{-3x^2+6y}{-6x+3y^2}=\frac{2y-x^2}{y^2-2x}$

$\boxed{k=2}$

Problem 3 [/7 marks]

Given that $xy=\cot(xy)$ and that derivative $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ can be written in the form $\dfrac{\mathrm{d}y}{\mathrm{d}x}=k\dfrac{y}{x}$, $k\in\mathbb{Z}$. Calculate the value of k.

$(xy)'=\left(\frac{\cos(xy)}{\sin(xy)}\right)'$

$y+xy'=\frac{-\sin^2(xy)(xy'+y)-\cos^2(xy)(xy'+y)}{\sin^2(xy)}$

$=\frac{-1}{\sin^2(xy)}(xy'+y)$

$y'\left(x+\frac{x}{\sin^2(xy)}\right)=y\left(-1-\frac{1}{\sin^2(xy)}\right)$

$y'=\dfrac{-y\left(1+\frac{1}{\sin^2(xy)}\right)}{x+\frac{x}{\sin^2(xy)}}$

$y'=\dfrac{-y\left(1+\frac{1}{\sin^2(xy)}\right)}{x\left(1+\frac{1}{\sin^2(xy)}\right)}=\dfrac{-y}{x}$

$\Rightarrow \boxed{k=-1}$