Christmas Examination

Thursday 18 Dec. 2025
Duration : one hour

Maths HL IB_2

Part 3

(8 Problems 48 marks)

Name : ____________________

A calculator **is allowed** for this third part

Problem 1

[/ 24 marks]

If two functions $f(x)$ and $g(x)$ are differentiable, then their product is differentiable and the two functions satisfy the product rule: $(f(x)g(x))' = f(x)g'(x) + g(x)f'(x)$.

In this question, you will meet examples of pairs of differentiable functions, $f(x)$ and $g(x)$, that also satisfy $(f(x)g(x))' = f'(x)g'(x)$.

In part (a), consider $f(x) = \dfrac{1}{(2-x)^2}$, where $x \in \mathbb{R}$, $x \neq 2$, and $g(x) = x^2$, where $x \in \mathbb{R}$.

(a) (i) Find an expression for $f'(x)$. [2]

(ii) Show that $f'(x)g'(x) = \dfrac{4x}{(2-x)^3}$. [2]

(iii) Show that $f(x)g'(x) + g(x)f'(x) = \dfrac{4x}{(2-x)^3}$. [4]

In parts (b) and (c), consider two non-constant functions, $f(x)$ and $g(x)$, where $f(x) > 0$ and $g(x) \neq g'(x)$.

(b) By rearranging the equation $f(x)g'(x) + g(x)f'(x) = f'(x)g'(x)$, show that $\dfrac{f'(x)}{f(x)} = \dfrac{g'(x)}{g'(x) - g(x)}$. [2]

(c) Hence, by integrating both sides of $\dfrac{f'(x)}{f(x)} = \dfrac{g'(x)}{g'(x) - g(x)}$, show that $f(x) = A\mathrm{e}^{\left(\int \frac{g'(x)}{g'(x)-g(x)}\,\mathrm{d}x\right)}$, where A is an arbitrary positive constant. [2]

The result from part (c) can be used to find pairs of functions, $f(x)$ and $g(x)$, which satisfy **both** of the following:

$$(f(x)g(x))' = f(x)g'(x) + g(x)f'(x) \text{ and } (f(x)g(x))' = f'(x)g'(x).$$

In parts (d) and (e), use the result in part (c) with $A = 1$.

(d) Consider $g(x) = x\mathrm{e}^x$.

Find $f(x)$ such that $f(x)$ and $g(x)$ satisfy the above two equations. [5]

(e) Consider $g(x) = \sin x + \cos x$.

Find $f(x)$ such that $f(x)$ and $g(x)$ satisfy the above two equations over the domain $0 < x < \pi$.

Give your answer in the form $f(x) = \sqrt{\mathrm{e}^x h(x)}$, where $h(x)$ is a function to be determined. [7]

Problem 2 [/ 21 marks]

This question asks you to find the probability of graphs of randomly generated quadratic functions having a specified number of x-intercepts.

In parts (a) – (f), consider quadratic functions, $f(x) = ax^2 + bx + c$, whose coefficients, a, b and c, are randomly generated in turn by rolling an unbiased six-sided die three times and reading off the value shown on the uppermost face of the die.

For example, rolling a 2, 3 and 5 in turn generates the quadratic function $f(x) = 2x^2 + 3x + 5$.

(a) Explain why there are 216 possible quadratic functions that can be generated using this method. [1]

(b) The set of coefficients, $a = 1$, $b = 4$ and $c = 4$, is randomly generated to form the quadratic function $f(x) = x^2 + 4x + 4$.

Verify that this graph of f has only one x-intercept. [2]

(c) By considering the discriminant, or otherwise, show that the probability of the graph of such a randomly generated quadratic function having only one x-intercept is $\frac{5}{216}$. [6]

Now consider randomly generated quadratic functions whose corresponding graphs have two **distinct** x-intercepts.

(d) By considering the discriminant, determine the set of possible values of ac. [3]

(e) (i) For the case where $ac = 1$, show that there are four quadratic functions whose corresponding graphs have two distinct x-intercepts. [1]

(ii) For the case where $ac = 2$, show that there are eight quadratic functions whose corresponding graphs have two distinct x-intercepts. [2]

Let p be the probability of the graph of such a randomly generated quadratic function having two distinct x-intercepts.

(f) Using the approach started in part (e), or otherwise, find the value of p. [6]

Problem 3 [/ 3 marks]

The Maclaurin series for $\tan(x)$ is $x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$

Use the Maclaurin series for $\tan(x)$ to find $\lim_{n\to\infty} \sin(\pi n^2 \tan\frac{\pi}{n^2})$