Christmas Examination

Maths HL IB$_2$

Part 2

(8 Problems 83 marks)

Monday 15 Dec. 2025

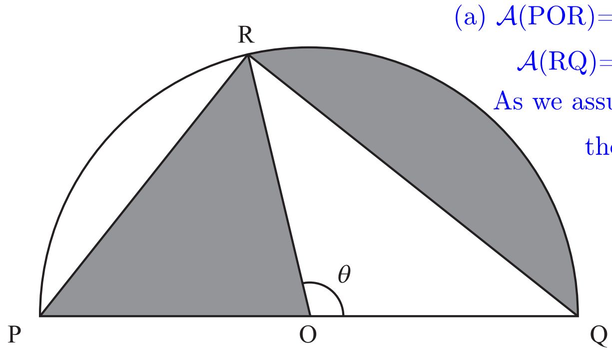
Duration : 2 hours

ANSWERS

Problem 1

[/ 6 marks]

The following diagram shows a semicircle with centre O and radius r. Points P, Q and R lie on the circumference of the circle, such that $PQ = 2r$ and $R\hat{O}Q = \theta$, where $0 < \theta < \pi$.



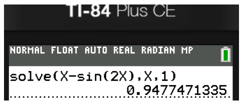
(a) $\mathcal{A}(\text{POR}) = \frac{r^2 \sin(\theta)}{2}$

$\mathcal{A}(\text{RQ}) = \frac{1}{2}r^2\theta - \frac{r^2\sin(\theta)}{2}$

As we assume $\mathcal{A}(\text{POR}) = \mathcal{A}(\text{RQ})$

then $2\frac{r^2\sin(\theta)}{2} = \frac{1}{2}r^2\theta$

$\Rightarrow \theta = 2\sin(\theta)$



(a) Given that the areas of the two shaded regions are equal, show that $\theta = 2\sin\theta$. [5]

(b) Hence determine the value of θ. $\boxed{\text{using a calculator (only !) : } \theta \cong 0.947 \text{ rad} \cong 54.3^o}$ [1]

Problem 2

[/ 7 marks]

(a) $e^{2it}(e^i + e^{3i}) = (\cos(2t) + i\sin(2t))(\cos(1) + i\sin(1) + \cos(3) + i\sin(3))$

$= (\cos(2t) + i\sin(2t))(\cos(1) + \cos(3) + i(\sin(1) + \sin(3)))$

$= (\cos(2t)(\cos(1) + \cos(3)) - (\sin(2t)(\sin(1) + \sin(3))$

$+ i((\sin(2t)(\cos(1) + \cos(3)) + \cos(2t)(\sin(1) + \sin(3))$

Hence $\text{im}(e^{2it}(e^i + e^{3i})) = \sin(2t)(\cos(1) + \cos(3)) + \cos(2t)(\sin(1) + \sin(3)$

$= \sin(2t)\cos(1) + \cos(2t)\sin(1) + \sin(2t)\cos(3) + \cos(2t)\sin(3)$

$= \sin(2t + 1) + \sin(2t + 3) = f(t) + g(t)$

(b) $e^i + e^{3i} = \cos(1) + \cos(3) + i(\sin(1) + \sin(3))$

- $r = |e^i + e^{3i}| = \sqrt{(\cos(1) + \cos(3))^2 + (\sin(1) + \sin(3))^2}$

  $= \sqrt{\cos^2(1) + \cos^2(3) + 2\cos(1)\cos(3) + \sin^2(1) + \sin^2(3) + 2\sin(1)\sin(3)}$

  $= \sqrt{\cos^2(1) + \sin^2(1) + \cos^2(3) + \sin^2(3) + 2(\cos(1)\cos(3) + \sin(1)\sin(3))}$

  $= \sqrt{1 + 1 + 2(\cos(3-1)} = \boxed{\sqrt{2 + 2(\cos(2)}}$ Notice : We also have $r = 2\cos(1) \cong 1.0806$

- $\theta = \arctan\left(\frac{\sin(1) + \sin(3)}{\cos(1) + \cos(3)}\right) = \boxed{2}$

(c) $h(t) = \text{Im}(e^{2it}\, 2\cos(1)e^{2i}) = \text{Im}(\, 2\cos(1)e^{i(2t+2)}) = \boxed{2\cos(1)\sin(2t + 2)}$

Problem 3 [/ 7 marks]

$f(x) = \frac{(2x+a)^3}{(x+5)^2}$ where $x \neq -5$ and $a \in \mathbb{R}^+$.

(a) $f'(x) = \frac{3(2x+a)^2 \cdot 2 \cdot (x+5)^2 - (2x+a)^3 \cdot 2 \cdot (x+5)}{((x+5)^2)^2} = \frac{6(2x+a)^2(x+5)^2 - 2(2x+a)^3 \cdot (x+5)}{(x+5)^4}$

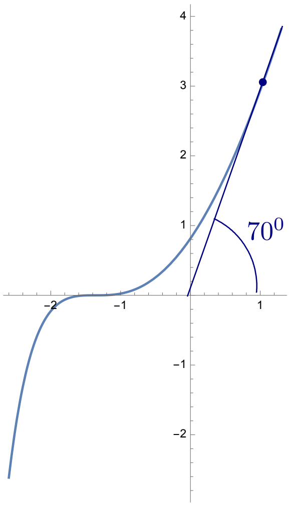
$= \frac{6(2x+a)^2(x+5) - 2(2x+a)^3}{(x+5)^3} = \frac{(2x+a)^2(6(x+5) - 2(2x+a))}{(x+5)^3} = \boxed{\frac{(2x+a)^2(2x+30-2a)}{(x+5)^3}}$

(b) The two possible values of a for having a tangent at $x = 1$ with an angle of 70^0

are solutions of $f'(x) = \tan(70) \cong 2.75$

Then $(2x+a)^2(2x+30-2a) = 2.75(x+5)^3$ when $x = 1$

that is : $(2+a)^2(32-2a) = 2.75 \cdot 6^3 = 593.5$ $\Rightarrow \boxed{a = 2.7306}$



Problem 4 [/ 21 marks]

$\frac{dy}{dx} = \frac{x^2+3y^2}{xy}$ where $x > 0, y > 0$ It is given that $y = 2$ when $x = 1$

(a) Euler's method : $y_{n+1} = y_n + F(x_n, y_n)\, h$ here with $x_0 = 1$ $y_0 = 2$ $h = 0.1$

$\Rightarrow$ $y_1 + = 2 + \frac{1^2 + 3 \times 2^2}{2} 0.1 = 2 + \frac{13}{2} \cdot \frac{1}{10} = 2 + \frac{13}{20} = \boxed{2.65}$

(b) $y' = \frac{x}{y} + 3\frac{y}{x}$ let be $q = \frac{y}{x}$ then $\frac{dq}{dx} = \frac{\frac{dy}{dx}x - y}{x^2} = \frac{1}{x}\frac{dy}{dx} - \frac{q}{x} \Rightarrow \frac{dy}{dx} = x\frac{dq}{dx} + q$

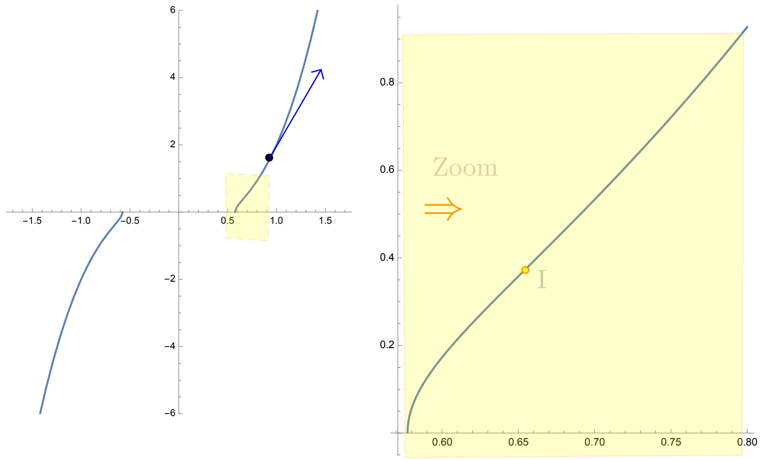
$\Rightarrow x\frac{dq}{dx} + q = \frac{1}{q} + 3q$ $\Rightarrow \frac{dq}{dx} = \frac{1}{xq} + \frac{2q}{x} = \frac{1+2q^2}{qx} \Leftrightarrow \int \frac{q\,dq}{1+2q^2} = \int \frac{dx}{x}$ $\Leftrightarrow \frac{1}{4}\ln(1+2q^2) = \ln(x) + c$

$\Rightarrow 1 + 2q^2 = Cx^4$ $\Rightarrow 2\frac{y^2}{x^2} = Cx^4 - 1$ $\Rightarrow$ $\frac{y^2}{x^2} = Cx^4 - \frac{1}{2} \Rightarrow \boxed{y(x) = \sqrt{Cx^6 - \frac{1}{2}x^2}}$

Finding C: $2 = \sqrt{C - \frac{1}{2}} \Leftrightarrow C - \frac{1}{2} = 4 \Leftrightarrow \boxed{C = \frac{9}{2}} \Leftrightarrow$ $y(x) = y(x) = \sqrt{\frac{9}{2}x^6 - \frac{1}{2}x^2} = x\sqrt{\frac{9x^4-1}{2}}$

(c) $y(1.1) \cong 1.1\sqrt{\frac{12.177}{2}} \cong \boxed{2.714}$

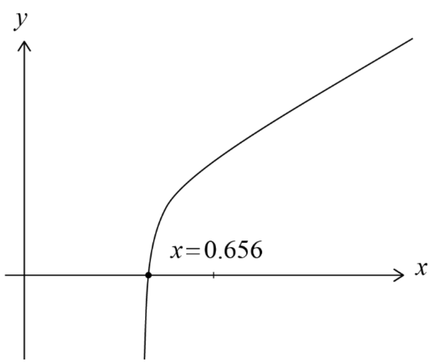
(d) $f''(x)>0$ for $x>1\in\mathrm{Dom}(f)$ $\Rightarrow$ *concave up* then $y_1<f(1.1)$.



e) As we can see on the figure below (zoom to the right),

there is a change of convexity for $x>0\in\mathrm{Dom}(f)$, near $x=0.7$.

We can investigate this property by looking to the graph of the second derivative:



That provides a better evaluation of the $x-$value of the inflection point I.

f) At this inflection point, we have $f''(x)=0$, and as w know $\frac{dy}{dx}=\frac{x^2+3y^2}{xy}$

To find $f''(x)$, we can derivate implicitly: (This step was **not** required !!)

$$f''(x)=\frac{(2x+6yy')xy-(x^2+3y^2)(y+xy')}{(xy)^2}$$

$$=\frac{2x^2y+6x^2y'-(x^2y+3y^3+x^3y'+3xy^2y')}{(xy)^2}$$

$$=\frac{2x^2y+6x^2y'-x^2y-3y^3-x^3y'-3xy^2y'}{(xy)^2}$$

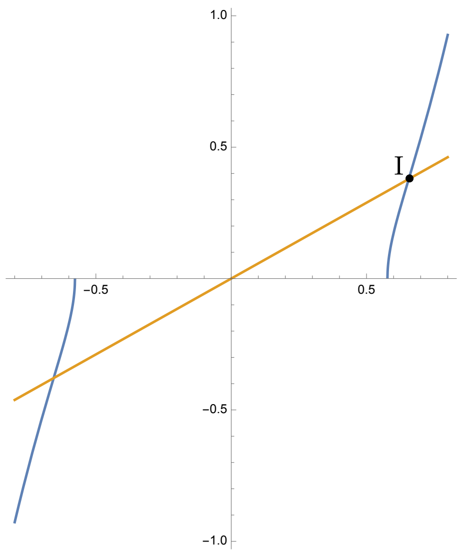
$$\ldots\quad\left[\text{replacing } y' \text{ by } \frac{x^2+3y^2}{xy}\right]$$

$$=\frac{-x^4+x^2y^2+6y^6}{x^2y^3}\qquad\text{(was given !)}$$

$$\Rightarrow -x^4+x^2y^2+6y^6=0$$

<u>3 methods</u> for finding m

(m is the gradient of the line, please see the figure)



EITHER

divides $-x^4+x^2y^2+6y^4(=0)$ through by y^4

$-\frac{x^4}{y^4}+\frac{x^2}{y^2}+6(=0)$

$m=\frac{y}{x}\Rightarrow -\frac{1}{m^4}+\frac{1}{m^2}+6(=0)$

$6m^4+m^2-1(=0)$

$(3m^2-1)(2m^2+1)=0$

$m=\pm\frac{1}{\sqrt{3}}\ \left(m^2=-\frac{1}{2}\right)$

OR

divides $-x^4+x^2y^2+6y^4(=0)$ through by x^2y^2

$-\frac{x^2}{y^2}+1+6\frac{y^2}{x^2}(=0)$

$m=\frac{y}{x}\Rightarrow -\frac{1}{m^2}+1+6m^2(=0)$

$6m^4+m^2-1(=0)$

$(3m^2-1)(2m^2+1)=0$

$m=\pm\frac{1}{\sqrt{3}}\ \left(m^2=-\frac{1}{2}\right)$

OR

attempts to factorize $-x^4+x^2y^2+6y^4(=0)$

$-(x^2-3y^2)(x^2+2y^2)(=0)$

attempts to solve their factorized equation

$\Rightarrow y=\pm\frac{1}{\sqrt{3}}x\ \left(y^2=-\frac{1}{2}x^2\right)$

Problem 5

[/ 20 marks]

Given $a=\frac{dv}{dt}=-(1+v)$, rewrite the differential equation:

$\frac{dv}{dt}+v=-1$

This is a first-order linear differential equation. The integrating factor is:

$e^{\int 1\,dt}=e^t$ ***M1***

Multiply through by e^t:

$\frac{d}{dt}(ve^t)=-e^t$

Integrate both sides:

$ve^t=-\int e^t\,dt=-e^t+c$

$v=-1+ce^{-t}$

Apply initial condition $v=v_0$ at $t=0$:

$v_0=-1+c \implies c=1+v_0.$

Thus, $v(t)=(1+v_0)e^{-t}-1.$

(i) At maximum displacement $s_{\max}$, $v(T)=0$. Using $v(t)=(1+v_0)e^{-t}-1$:

$v(T)=(1+v_0)e^{-T}-1=0$

$(1+v_0)e^{-T}=1 \implies 1+v_0=e^T$

(ii) Since $\frac{ds}{dt} = v(t) = (1 + v_0)e^{-t} - 1$, integrate to find $s(t)$:

$s = \int((1 + v_0)e^{-t} - 1)\, dt = -(1 + v_0)e^{-t} - t + c$.

At $t = 0$, $s = 0$:

$0 = -(1 + v_0)e^0 + c = -(1 + v_0) + c \implies c = 1 + v_0$

$s(t) = (1 + v_0) - (1 + v_0)e^{-t} - t$.

At $t = T$, $s = s_{\max}$, and from (b)(i), $e^T = 1 + v_0 \implies e^{-T} = \frac{1}{1+v_0}$:

$s_{\max} = (1 + v_0) - (1 + v_0)e^{-T} - T = (1 + v_0) - (1 + v_0) \cdot \frac{1}{1+v_0} - \ln(1 + v_0) = (1 + v_0) - 1 - \ln(1 + v_0)$

c.

Given $v(T - k) = (1 + v_0)e^{-(T-k)} - 1$, and from (b)(i), $1 + v_0 = e^T$:

$v(T - k) = (1 + v_0)e^{-T}e^k - 1 = e^T \cdot e^{-T}e^k - 1 = e^k - 1$

d.

For $v(T + k) = (1 + v_0)e^{-(T+k)} - 1$:

Using $1 + v_0 = e^T$:

$v(T + k) = (1 + v_0)e^{-T}e^{-k} - 1 = e^T \cdot e^{-T}e^{-k} - 1 = e^{-k} - 1$

e.

Using results from (c) and (d):

$v(T - k) + v(T + k) = (e^k - 1) + (e^{-k} - 1) = e^k + e^{-k} - 2$

Rewrite: $e^k + e^{-k} - 2 = \frac{e^{2k}+1-2e^k}{e^k} = \frac{(e^k-1)^2}{e^k}$

Since $(e^k - 1)^2 \geq 0$ and $e^k > 0$, $\frac{(e^k-1)^2}{e^k} \geq 0$. Thus, $v(T - k) + v(T + k) \geq 0$.

Problem 6 [/ 6 marks]

$\frac{dy}{dx} = \frac{2x}{x^2 + y}$ the solution passes through the point (1,0)

(a) Euler's method : $y_{n+1} = y_n + F(x_n, y_n)\,h$ here with $x_0 = 1$ $y_0 = 0$ $h = 0.25$ four steps

step 1: $x_1 = 1.25$ and $y_1 = 0 + \frac{2 \times 1}{1^2 + 0} 0.25 = 0 + 2 \cdot \frac{1}{4} = 0.5$

step 2: $x_2 = 1.5$ and $y_2 = 0.5 + \frac{2 \times 1.25}{1.25^2 + 0.5} 0.25 = 1.5 + \frac{2.5}{3.06} \cdot \frac{1}{4} = 1.5$

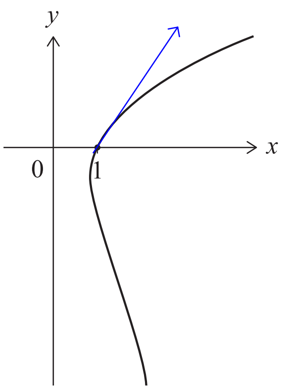
step 3: $x_3 = 1.75$ and $y_3 = 1.5 + \frac{2 \times 1.5}{1.5^2 + 1.5} 0.25 = 1.0487$

step 4: $x_3 = 2$ and $y_4 = 1.0487 + \frac{2 \times 1.0487}{1.0487^2 + 1.75} 0.25 = \boxed{1.2615}$

v

n	x	y
0	1.	0.
1	1.25	0.5
2	1.5	0.80303
3	1.75	1.04869
4	2.	1.26152

b) i) We can see on the figure below that Euler's method gives here an overestimation.



ii) That is because at the initial position, the gradient is *positive*.

Problem 7

[/ 8 marks]

(a) State two conditions required for X to be modeled by a binomial distribution.

Fixed number of trials

Each trial has *two* possible outcomes

The outcome of each trial is *independent* of all the others (*constant* probability of success)

(b) $1900 \times 0.37 = \boxed{703 \text{ people}}$

(c) (i) Let D be the number of people who will ride *Daifong*

$\mathrm{P}(D = 712) = \boxed{0.0172556...}$

(ii) $P(D \leqslant 712) = \underline{0.674739...}$

$P(D \leqslant 683) = 0.177146...$

$\mathrm{P}(684 \leqslant D \leqslant 712) = \boxed{0.497593...}$

That would be interesting to continue this question, looking at Paper 2 May 2025 tz3

(d) $\mathrm{P}(\leqslant D \leqslant 692 | 684 \leqslant D \leqslant 712) = \frac{\mathrm{P}(684 \leqslant D \leqslant 692)}{\mathrm{P}(684 \leqslant D \leqslant 712)} = \frac{0.132318}{0.497593} = \boxed{0.266}$

Problem 8

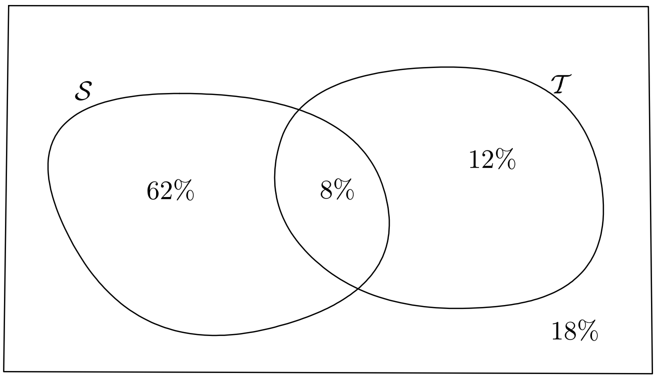
[/ 8 marks]

a) See the diagram

b) $\mathrm{P}(\mathcal{S} \cap \mathcal{T}^*) = 12\%$

c) $\mathrm{P}(\mathcal{G} \cap \mathcal{T}) = P(\mathcal{G}) P(\mathcal{T} | \mathcal{G})$
$= 0.48 \times 0.25 = 12\%$

d) $\mathrm{P}(\mathcal{T} | \mathcal{G}) \neq \mathrm{P}(\mathcal{T}) \Rightarrow$ *not* independent.
$\mathrm{P}(\mathcal{T} \cap \mathcal{G}) \neq P(\mathcal{T}) P(G) \Rightarrow$ *not* indep!



Bonus

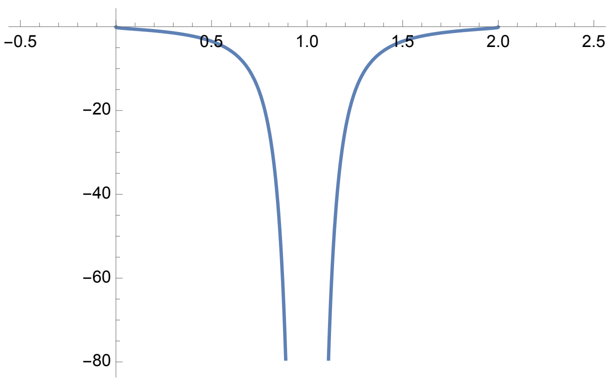
[+ 7]

Let us consider the function

$$f(x) = \frac{1}{\ln(2x - x^2)}$$

i) The *domain* of f is $\mathbb{R} \backslash \{1\}$

ii) The curve of equation $y = f(x)$ is shown ...



iii) The equation of the *vertical* asymptote: $\boxed{x = 1}$

iv) The equation of the horizontal asymptote: $\boxed{y = 0}$