Christmas Examination

Monday 15 Dec. 2025

Duration : 2 hours

Maths HL IB_2

Part 2

(8 Problems 83 marks)

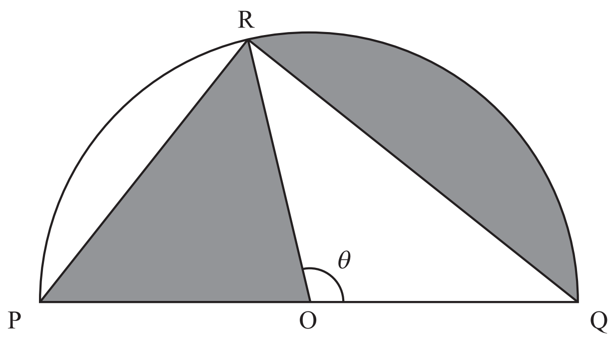
Name : ____________________

A calculator **is allowed** for this second part

Problem 1

[/ 6 marks]

The following diagram shows a semicircle with centre O and radius r. Points P, Q and R lie on the circumference of the circle, such that $\mathrm{PQ} = 2r$ and $\mathrm{R\hat{O}Q} = \theta$, where $0 < \theta < \pi$.



(a) Given that the areas of the two shaded regions are equal, show that $\theta = 2\sin\theta$. [5]

(b) Hence determine the value of θ. [1]

Problem 2

[/ 7 marks]

Consider the functions f, g and h defined as follows for $t \in \mathbb{R}$.

$$f(t) = \sin(2t+1)$$

$$g(t) = \sin(2t+3)$$

$$h(t) = f(t) + g(t)$$

(a) Show that $h(t) = \mathrm{Im}\left(\mathrm{e}^{2ti}(\mathrm{e}^{i} + \mathrm{e}^{3i})\right)$. [2]

(b) Write $\mathrm{e}^{i} + \mathrm{e}^{3i}$ in the form $r\mathrm{e}^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [2]

(c) Hence or otherwise, write $h(t)$ in the form $p\sin(2t+q)$, where $p > 0$ and $0 < q < 2\pi$. [3]

Problem 3 [/ 7 marks]

Consider the function $f(x)=\frac{(2x+a)^3}{(x+5)^2}$, where $x \neq -5$ and $a \in \mathbb{R}^+$.

(a) Find an expression for $f'(x)$, in terms of a. [3]

When $x=1$, the tangent to the graph of f makes an angle of $70°$ to the horizontal.

(b) Find the two possible values of a. [4]

Problem 4 [/ 21 marks]

Consider the differential equation $\frac{dy}{dx}=\frac{x^2+3y^2}{xy}$, where $x>0$, $y>0$.

It is given that $y=2$ when $x=1$.

(a) Use Euler's method with step length 0.1 to find an approximate value of y when $x=1.1$. [2]

(b) By solving the differential equation, show that $y=x\sqrt{\frac{9x^4-1}{2}}$. [8]

(c) Find the value of y when $x=1.1$. [1]

(d) With reference to the concavity of the graph of $y=x\sqrt{\frac{9x^4-1}{2}}$ for $1 \le x \le 1.1$, explain why the value of y found in part (c) is greater than the approximate value of y found in part (a). [2]

The graph of $y=x\sqrt{\frac{9x^4-1}{2}}$ for $\frac{\sqrt{3}}{3}<x<1$ has a point of inflexion at the point P.

(e) By sketching the graph of an appropriate derivative of y, determine the x-coordinate of P. [2]

It can be shown that $\frac{d^2y}{dx^2}=\frac{-x^4+x^2y^2+6y^4}{x^2y^3}$, where $x>0$, $y>0$.

(f) Use this expression for $\frac{d^2y}{dx^2}$ to show that point P lies on the straight line $y=mx$ where the exact value of m is to be determined. [6]

Problem 5 [/ 20 marks]

The acceleration, $a\,\mathrm{ms}^{-2}$, of a particle moving in a horizontal line at time t seconds, $t \geq 0$, is given by $a = -(1+v)$ where $v\,\mathrm{ms}^{-1}$ is the particle's velocity and $v > -1$.

At $t = 0$, the particle is at a fixed origin O and has initial velocity $v_0\,\mathrm{ms}^{-1}$.

(a) By solving an appropriate differential equation, show that the particle's velocity at time t is given by $v(t) = (1 + v_0)\mathrm{e}^{-t} - 1$. [6]

(b) Initially at O, the particle moves in the positive direction until it reaches its maximum displacement from O. The particle then returns to O.

Let s metres represent the particle's displacement from O and $s_{\max}$ its maximum displacement from O.

(i) Show that the time T taken for the particle to reach $s_{\max}$ satisfies the equation $\mathrm{e}^{T} = 1 + v_0$.

(ii) By solving an appropriate differential equation and using the result from part (b) (i), find an expression for $s_{\max}$ in terms of v_0. [7]

Let $v(T-k)$ represent the particle's velocity k seconds before it reaches $s_{\max}$, where

$$v(T-k) = (1 + v_0)\,\mathrm{e}^{-(T-k)} - 1\,.$$

(c) By using the result to part (b) (i), show that $v(T-k) = \mathrm{e}^{k} - 1$. [2]

Similarly, let $v(T+k)$ represent the particle's velocity k seconds after it reaches $s_{\max}$.

(d) Deduce a similar expression for $v(T+k)$ in terms of k. [2]

(e) Hence, show that $v(T-k) + v(T+k) \geq 0$. [3]

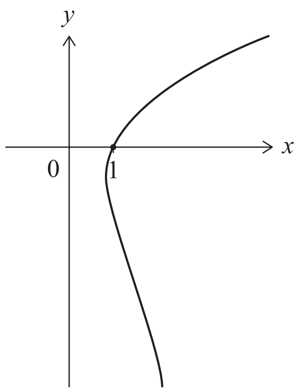
Problem 6 [/ 6 marks]

Consider the differential equation $\frac{dy}{dx} = \frac{2x}{x^2 + y}$.

The solution curve passes through the point $(1, 0)$.

(a) Use Euler's method with a step value of 0.25 to estimate the value of y when $x = 2$. [3]

Part of the solution curve is shown in the following diagram.



(b) (i) Determine whether your answer to part (a) is an overestimate or an underestimate, justifying your answer.

(ii) Justify why the use of Euler's method starting at $(1, 0)$ does not lead to an estimate of the negative value of y when $x = 2$. [3]

Problem 7 [/ 8 marks]

(a) State two conditions required for X to be modelled by a binomial distribution. [2]

A water theme park has two rides: *Daifong* and *Torbellino*. Each visitor's decision to ride on either *Daifong* or *Torbellino* is made independently of any other person.

From previous records, it is expected that 37% of the visitors on any particular day will ride *Daifong*.

On Saturday, 1900 people will visit the theme park.

(b) Find the number of people that are expected to ride *Daifong*. [2]

(c) Find the probability that

(i) 712 people will ride *Daifong*;

(ii) between 684 and 712 people, inclusive, will ride *Daifong*. [4]

Problem 8 [/ 8 marks]

At a school, 70% of the students play a sport and 20% of the students are involved in theatre. 18% of the students do neither activity.

A student is selected at random.

(a) Find the probability that the student plays a sport and is involved in theatre. [2]

(b) Find the probability that the student is involved in theatre, but does not play a sport. [2]

At the school 48% of the students are girls, and 25% of the girls are involved in theatre.

A student is selected at random. Let G be the event "the student is a girl" and let T be the event "the student is involved in theatre".

(c) Find $\mathrm{P}(G \cap T)$. [2]

(d) Determine if the events G and T are independent. Justify your answer. [2]

Bonus [+ 7]

Let us consider the function

$$f(x) = \frac{1}{\ln(2x - x^2)}$$

i) Give the domain of f

ii) Draw the curve of equarion $y = f(x)$

iii) Give the equation of the vertical asymptote.

iv) Give the equation of the horizontal asymptote.