Christmas Examination

Maths HL IB$_2$

Part 1

(8 IB questions 106 marks)

Monday 15 Dec.2025

Duration : 2 hours

ANSWERS

A calculator is not allowed for this fisrt part

Problem 1 [/ 19 marks]

(a) Solve $z^2 = -1 - \sqrt{3}i$, giving your answer in the form $z = r(\cos\theta + i\sin\theta)$

the modulus of $-1-\sqrt{3}i$ is 2, and its argument is $\arctan(\sqrt{3}) = \frac{\pi}{3} + \pi = \frac{4\pi}{3} + 2k\pi$

therefore the modulus of z is $\sqrt{2}$ and its argument is $\frac{2\pi}{3} + k\pi$, then $\boxed{z = \begin{array}{l} \sqrt{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) \\ \sqrt{2}\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) \end{array}}$

Verification : If $z = \pm\sqrt{2}\left(\left(-\frac{1}{2}\right) + i\frac{\sqrt{3}}{2}\right) = \pm\frac{\sqrt{2}}{2}(-1+\sqrt{3}i)$

then $z^2 = \frac{1}{2}(1 - 3 - 2\sqrt{3}i) = -1 - \sqrt{3}i$ ok!

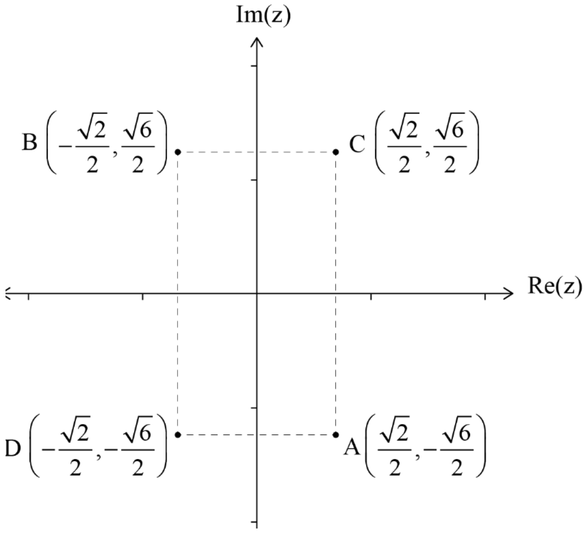
(b) Let z_1 and z_2 be the square roots of $-1-\sqrt{3}i$

$z_1 = \frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i \qquad z_2 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i$

Let z_3 and z_4 be the square roots of $-1-\sqrt{3}i$

$z_3 = \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i \qquad z_4 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2}i$

(c)



The area of the rectangle is : $\mathcal{A} = \text{base} \times \text{hight} = \sqrt{2} \times \sqrt{6} = \boxed{2\sqrt{3}}$

(d) $1\left(\frac{1}{w}\right)^4 + 2\left(\frac{1}{w}\right)^2 + 4 = 0 \Leftrightarrow \frac{1 + 2w^2 + 4w^4}{w} = 0$ then w root of $4w^4 + 2w^2 + 1 = 0$

$\boxed{p = 4, q = 2, r = 1}$

(e) $\frac{1}{z_1} = \frac{z_1^*}{|z_1|^2} = \frac{z_1^*}{2}$ same for the other k_j then $\mathcal{A}' = \frac{\text{base}}{2} \times \frac{\text{hight}}{2} = \frac{\mathcal{A}}{4} = \boxed{\frac{\sqrt{3}}{2}}$

Problem 2

[/ 6 marks]

Use l'Hôpital's rule to find $\lim_{x\to 0}\dfrac{\sec^4 x-\cos^2 x}{x^4-x^2}$.

$$l=\lim_{x\to 0}\frac{4\sec^4 x\tan x+2\sin x\cos x}{4x^3-2x}=\lim_{x\to 0}\frac{16\sec^4 x\tan^2 x+4\sec^6 x-2\sin^2 x+2\cos^2 x}{12x^2-2}=\boxed{-3}$$

Problem 3

[/ 7 marks]

Consider the function $f(x)=\sqrt{x^2\ln(x)+4-x^2}$, where $x\in\mathbb{R}^+$

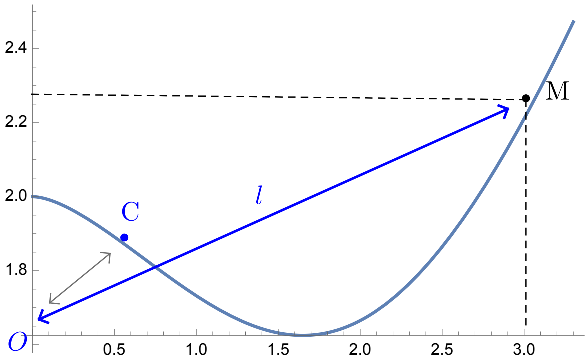
a) The distance l between the origin and any point on the graph of f is given by

$l=\text{dist}(O,M)$

where M has coordinates $(x,f(x))$

then $l(x)=\sqrt{x^2+f(x)^2}$

$=\sqrt{x^2\ln(x)+4}$



b) To find the closest point C to the origin, we derivate $l(x)$:

$$l'(x)=\frac{x^2\frac{1}{x}+2x\ln(x)}{2\sqrt{x^2\ln(x)+4}}$$

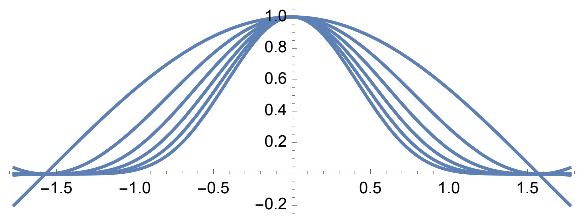
$l'(x)=0$ if $x(1+2\ln(x))=0\Rightarrow \ln(x)=-\frac{1}{2}\Rightarrow x=e^{-\frac{1}{2}}$

$$\Rightarrow \boxed{\text{C}:(e^{-\frac{1}{2}},\sqrt{-\tfrac{1}{2e}+4-\tfrac{1}{e}})}\quad\cong(0.607,1.86)$$

Problem 4

[/ 19 marks]

Consider the family of functions $f_n(x)=\cos^n(x)$, where $x\in\mathbb{R}$ and $n\in\mathbb{N}$.



(a)
$$\begin{aligned}\int\cos^n(x)dx&=\cos^{(n-1)}(x)\sin(x)-\int(n-1)\cos^{(n-2)}(x)(-\sin(x))\sin(x)dx\\&=\cos^{(n-1)}(x)\sin(x)+(n-1)\int\cos^{(n-2)}(x)\sin^2(x)dx\\&=\cos^{(n-1)}(x)\sin(x)+(n-1)\int\cos^{(n-2)}(x)(1-\cos^2(x))dx\\&=\cos^{(n-1)}(x)\sin(x)+(n-1)\int\cos^{(n-2)}(x)dx-(n-1)\int\cos^n(x)dx\qquad(\text{for } n>1)\end{aligned}$$

(b) Hence : $(1+n-1)\int f_n(x)dx = f_{(n-1)}(x)\sin(x) + (n-1)\int f_{(n-2)}(x)dx$

$$\Rightarrow \boxed{\int f_n(x)dx = \frac{\sin(x)}{n} f_{(n-1)}(x) + \frac{(n-1)}{n}\int f_{(n-2)}(x)dx}$$

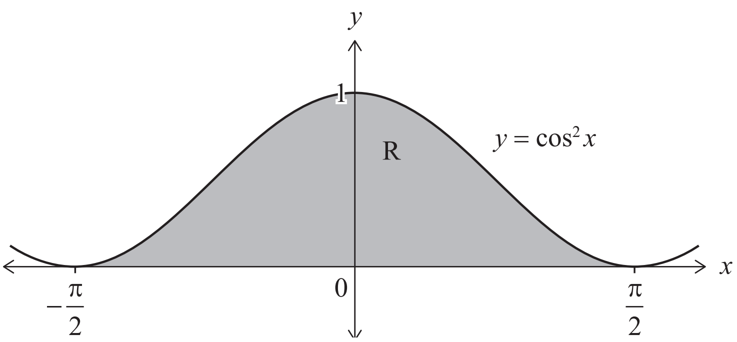
(c) Therefore $\int \cos^4 dx = \int f_4(x)dx = \frac{\sin(x)}{4}\cos^3(x) + \frac{(4-1)}{4}\int \cos^2 dx$

$$= \tfrac{\sin(x)}{4}\cos^3(x) + \tfrac{3}{4}\left(\tfrac{x}{2} + \tfrac{1}{2}\sin(x)\cos(x)\right)$$

$$= \boxed{\tfrac{1}{4}\sin(x)\cos^3(x) + \tfrac{3}{16}\sin(2x) + \tfrac{3x}{8}}$$

$$\left(=\tfrac{1}{32}\,\mathrm{Sin}(4x) + \tfrac{1}{4}\sin(2x) + \tfrac{3x}{8}\right)$$

(d) The volume of the solid of revolution formed by the rotation of R around the $x-$axis



is : $V = \pi\int (f_2(x))^2 dx = \pi\int \cos^4 dx = \frac{\pi}{4}\left[\sin(x)\cos^3(x) + \frac{3}{4}\sin(2x) + \frac{3x}{2}\right]_{-\frac{\pi}{2}}^{-\frac{\pi}{2}} = \boxed{\frac{3\pi^2}{8}}$

(e) i) The Maclaurin series of $f_n(x)$ is the binomial expansion of $\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)\right)^n$

$$= (1)^n + n(1)^{n-1}(-\tfrac{x^2}{2} + \tfrac{x^4}{24} + O(x^6)) + \tfrac{n(n-1)}{2}1^{n-2}(-\tfrac{x^2}{2} + \tfrac{x^4}{24} + O(x^6))^2 + \tfrac{n(n-1)(n-2)}{2}(-\tfrac{x^2}{2} + \ldots)^3$$

$$= 1 - n\tfrac{x^2}{2} + \tfrac{n\,x^4}{24} + \tfrac{n\,x^6}{720}\cdots + \tfrac{n(n-1)}{2}\tfrac{x^4}{4} + \tfrac{n(n-1)}{2}\tfrac{2x^{46}}{48} + \cdots$$

$$\doteq 1 - n\tfrac{x^2}{2} + \left(\tfrac{n}{24} + \tfrac{n(n-1)}{8}\right)x^4 + \left(\tfrac{n}{720} + \tfrac{n(n-1)}{48} + \tfrac{n(n-1)(n-2)}{2}\right)\cdot x^6 + \cdots$$

$$\doteq \boxed{1 - n\tfrac{x^2}{2}} + \tfrac{1}{24}(3n^2 - 2n)x^4 + \ldots.$$

ii) Hence $\lim\limits_{x\to 0}\frac{f_n(x)-1}{x^2} = \lim\limits_{x\to 0}\frac{1 - n\frac{x^2}{2} - 1}{x^2}\ (+O(x^4) = \lim\limits_{x\to 0}\frac{-n\frac{x^2}{2}}{x^2} = \boxed{-\frac{n}{2}}$

Problem 5 [/ 17 marks]

The function f is defined by $f(x)=4^x$, where $x\in\mathbb{R}$.

(a) $f^{-1}(8)=\log_4(8)=\log_4(\sqrt{4}\times 4)=\boxed{\frac{3}{2}}$

(b) $g(x)=1+\log_2(x)$

i) $g^{-1}(x)=2^{x-2}$

ii) $g^{-1}(2x+2)=2^{2x}=4^x=f(x)$

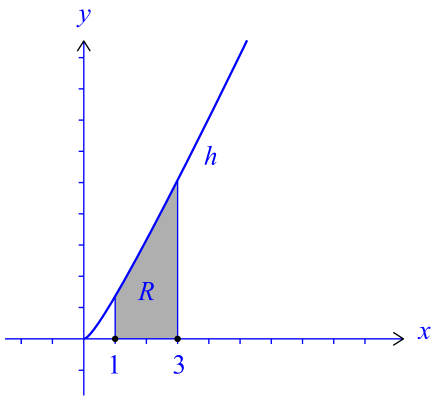
therefore the sequence of transformation is : $-$horizontal translation of two units left

$-$horizontal contraction of factor 2

(c) $(f\circ g)(x)=4^{1+\log_2(x)}=4^1\times 4^{\log_2(x)}=4\times(2^{\log_2(x)})^2=4x^2$

(d) i) $2x-1+\frac{1}{2x+1}=\frac{4x^2}{2x+1}$

ii) $$R=\int_1^3\frac{4x^2}{2x+1}dx$$
$$=\int_1^3\left(2x-1+\frac{1}{2x+1}\right)dx$$
$$=\left[x^2-x-\frac{1}{2}\ln(2x+1)\right]_1^3$$
$$=\left(9-3-\frac{1}{2}\ln(7)\right)-\left(1-1-\frac{1}{2}\ln(3)\right)$$
$$=\boxed{6+\frac{1}{2}\ln\left(\frac{7}{3}\right)}$$



Problem 6 [/ 20 marks]

Question on kinematics.

Please see Paper II Question 5

Problem 7 [/ 5 marks]

Box 1 contains 5 red balls and 2 white balls.
Box 2 contains 4 red balls and 3 white balls.

(a) A box is chosen at random and a ball is drawn. Find the probability that the ball is red. [3]

Let A be the event that “box 1 is chosen” and let R be the event that “a red ball is drawn”.

(b) Determine whether events A and R are independent. [2]

a) $\mathrm{P(red)}=\frac{1}{2}\times\frac{5}{7}+\frac{1}{2}\times\frac{4}{7}=\frac{9}{14}$

b) $\mathrm{P(red|box1)}=\frac{\mathrm{P(red\cap box1)}}{\mathrm{P(box1)}}=\frac{\frac{1}{2}\times\frac{5}{7}}{\frac{1}{2}}=\frac{5}{7}\neq\frac{9}{14}\Rightarrow$ <u>not</u> independant.

Problem 8 [/ 13 marks]

A discrete random variable, X, has the following probability distribution, where $a>0$ and k is a constant.

x	0	a	$2a$	$3a$
$\mathrm{P}(X=x)$	k	$3k^2$	$2k^2$	k^2

(a) Show that $k=\dfrac{1}{3}$. [5]

(b) Find $\mathrm{P}(X<3a)$. [2]

(c) Find $\mathrm{P}(X\geq a|X<3a)$. [3]

(d) Given that $\mathrm{E}(X)=20$, find the value of a. [3]

(a) $k+3k^2+2k^2+k^2=1 \quad \Rightarrow 6k^2+k-1=0 \qquad \Delta=25 \quad k=\frac{-1\pm5}{12}$ as $k>0$: $k=\frac{4}{12}=\frac{1}{3}$

(b) $\mathrm{P(X<3a)}=k+5k^2=\boxed{\frac{8}{9}}$

(c) $\mathrm{P}(X\geqslant 2a|X<3a)=\frac{P(X\geqslant 2a\cap X<3a)}{P(X<3a)}=\frac{P(X=2a)}{P(X<3a)}=\frac{2k^2}{k+5k^2}=\frac{\frac{2}{9}}{\frac{8}{9}}=\boxed{\frac{1}{4}}$

(d) $\mathrm{E}(X)=k\times0+3k^2a+2k^2 2a+k^2 3a = 10ak^2=20$ with $k=\frac{1}{3}$ then $\boxed{a=18}$