Christmas Examination

Maths HL IB_2

Part 1

(8 IB questions 106 marks)

Monday 15 Dec.2025
Duration : 2 hours

Name: ___________

A calculator is not allowed for this fisrt part

Problem 1

[/ 19 marks]

(a) Solve $z^2 = -1 - \sqrt{3}\,\mathrm{i}$, giving your answers in the form $z = r(\cos\theta + \mathrm{i}\sin\theta)$. [4]

Let z_1 and z_2 be the square roots of $-1-\sqrt{3}\,\mathrm{i}$, where $\mathrm{Re}(z_1) > 0$.

Let z_3 and z_4 be the square roots of $-1+\sqrt{3}\,\mathrm{i}$, where $\mathrm{Re}(z_3) > 0$.

(b) Expressing your answers in the form $z = a + b\mathrm{i}$, where $a, b \in \mathbb{R}$,

(i) find z_1 and z_2;

(ii) deduce z_3 and z_4. [4]

The four roots z_1, z_2, z_3 and z_4 are represented by the points A, B, C and D respectively on an Argand diagram.

(c) (i) Plot the points A, B, C and D on an Argand diagram.

(ii) Find the area of the polygon formed by these four points. [4]

The four roots z_1, z_2, z_3 and z_4 satisfy the equation $z^4 + 2z^2 + 4 = 0$.

The four roots $\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}$ and $\frac{1}{z_4}$ satisfy the equation $pw^4 + qw^2 + r = 0$ where $p, q, r \in \mathbb{Z}$.

(d) Find the value of p, q and r. [3]

The four roots $\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}$ and $\frac{1}{z_4}$ are represented by the points E, F, G and H respectively on an Argand diagram.

(e) (i) Find $\frac{1}{z_1}$ in the form $z = a + b\mathrm{i}$, where $a, b \in \mathbb{R}$.

(ii) Hence, deduce the area of the polygon formed by these four points. [4]

Problem 2 [/ 6 marks]

Use l'Hôpital's rule to find $\lim_{x\to 0}\dfrac{\sec^4 x-\cos^2 x}{x^4-x^2}$.

Problem 3 [/ 7 marks]

Consider the function $f(x)=\sqrt{x^2\ln x+4-x^2}$, where $x\in\mathbb{R},\ x>0$.

(a) Show that the distance, l, between the origin and any point on the graph of f is given by $l=\sqrt{x^2\ln x+4}$. [1]

(b) Hence, find the x-coordinate of the point on the graph of f which is closest to the origin. [6]

Problem 4 [/ 19 marks]

Consider the family of functions $f_n(x)=\cos^n x$, where $x\in\mathbb{R}$ and $n\in\mathbb{N}$.

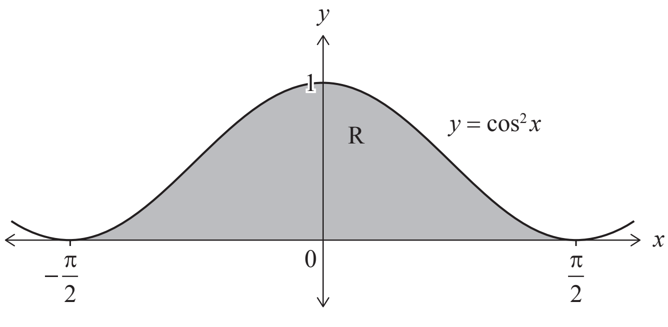
(a) By writing $\cos^n x$ as $\cos^{n-1}x\cos x$, show that

$$\int\cos^n x\,dx=\cos^{n-1}x\sin x+(n-1)\int\cos^{n-2}x\,dx-(n-1)\int\cos^n x\,dx \text{ for } n>1.$$ [4]

(b) Hence, show that $\int f_n(x)dx=\dfrac{1}{n}\cos^{n-1}x\sin x+\dfrac{n-1}{n}\int f_{n-2}(x)dx$ for $n>1$. [2]

(c) Hence, find an expression for $\int\cos^4 x\,dx$, giving your answer in the form $p\cos^3x\sin x+q\cos x\sin x+rx+c$ where $p,q,r\in\mathbb{Q}^+$. [4]

The region R is enclosed by the graph of $y=\cos^2 x$ and the x-axis where $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$, as shown in the following diagram.



The region R is rotated by 2π radians around the x-axis to form a solid of revolution.

(d) Find the volume of the solid. [4]

(e) (i) Find the Maclaurin series of $f_n(x)$ up to the term in x^2.

(ii) Hence or otherwise, find $\lim_{x\to 0}\dfrac{f_n(x)-1}{x^2}$ in terms of n. [5]

Problem 5 [/ 17 marks]

The function f is defined by $f(x) = 4^x$, where $x \in \mathbb{R}$.

(a) Find $f^{-1}(8)$. Express your answer in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$. [3]

The function g is defined by $g(x) = 1 + \log_2 x$, where $x \in \mathbb{R}^+$.

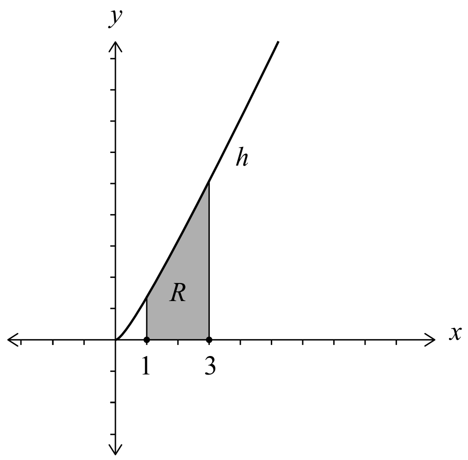
(b) (i) Find an expression for $g^{-1}(x)$.

(ii) Describe a sequence of transformations that transforms the graph of $y = g^{-1}(x)$ to the graph of $y = f(x)$. [4]

(c) Show that $(f \circ g)(x) = 4x^2$. [3]

The function h is defined by $h(x) = \frac{4x^2}{2x+1}, x \neq -\frac{1}{2}$.

The following diagram shows part of the graph of h. Let R be the region enclosed by the graph of h and the x-axis, between the lines $x = 1$ and $x = 3$.



(d) (i) Show that $2x - 1 + \frac{1}{2x+1} = \frac{4x^2}{2x+1}$.

(ii) Hence or otherwise, find the area of R, giving your answer in the form $p + q\ln r$, where $p, q, r \in \mathbb{Q}^+$. [7]

Problem 6 [/ 20 marks]

The acceleration, $a\,\text{ms}^{-2}$, of a particle moving in a horizontal line at time t seconds, $t \geq 0$, is given by $a = -(1+v)$ where $v\,\text{ms}^{-1}$ is the particle's velocity and $v > -1$.

At $t = 0$, the particle is at a fixed origin O and has initial velocity $v_0\,\text{ms}^{-1}$.

(a) By solving an appropriate differential equation, show that the particle's velocity at time t is given by $v(t) = (1 + v_0)\,\text{e}^{-t} - 1$. [6]

(b) Initially at O, the particle moves in the positive direction until it reaches its maximum displacement from O. The particle then returns to O.

Let s metres represent the particle's displacement from O and $s_{\max}$ its maximum displacement from O.

(i) Show that the time T taken for the particle to reach $s_{\max}$ satisfies the equation $\text{e}^{T} = 1 + v_0$.

(ii) By solving an appropriate differential equation and using the result from part (b) (i), find an expression for $s_{\max}$ in terms of v_0. [7]

Let $v(T-k)$ represent the particle's velocity k seconds before it reaches $s_{\max}$, where

$$v(T-k) = (1 + v_0)\,\text{e}^{-(T-k)} - 1\,.$$

(c) By using the result to part (b) (i), show that $v(T-k) = \text{e}^{k} - 1$. [2]

Similarly, let $v(T+k)$ represent the particle's velocity k seconds after it reaches $s_{\max}$.

(d) Deduce a similar expression for $v(T+k)$ in terms of k. [2]

(e) Hence, show that $v(T-k) + v(T+k) \geq 0$. [3]

Problem 7

[/ 5 marks]

Box 1 contains 5 red balls and 2 white balls.
Box 2 contains 4 red balls and 3 white balls.

(a) A box is chosen at random and a ball is drawn. Find the probability that the ball is red. [3]

Let A be the event that "box 1 is chosen" and let R be the event that "a red ball is drawn".

(b) Determine whether events A and R are independent. [2]

Problem 8

[/ 13 marks]

A discrete random variable, X, has the following probability distribution, where $a > 0$ and k is a constant.

x	0	a	$2a$	$3a$
$\mathrm{P}(X = x)$	k	$3k^2$	$2k^2$	k^2

(a) Show that $k = \frac{1}{3}$. [5]

(b) Find $\mathrm{P}(X < 3a)$. [2]

(c) Find $\mathrm{P}(X \geq a | X < 3a)$. [3]

(d) Given that $\mathrm{E}(X) = 20$, find the value of a. [3]