I.B. Mathematics HL Core: Vector Geometry 02

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Question 1

Let a line M and a plane P be given by

$$M: \quad \frac{x-4}{1} = \frac{y-6}{2} = \frac{z-2}{3}$$

$$P: \quad 3x + 4y - z = 2$$

a. Find the equation of another line L, which passes through the point (4, -2, 1) and is parallel to the vector $2\vec{i} - \vec{j} + 2\vec{k}$.

b. Find the coordinates of the point of intersection of the line M and the plane P.

c. Calculate the acute angle between M and P, giving your answer correct to the nearest one-tenth of a degree.

d. Show that the line L is parallel to the plane P.

e. Calculate the distance between the line L and the plane P.

f. Show that the lines L and M do not intersect.

g. Calculate the distance between the lines L and M.

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Solution to question 1

a. A vector equation of a line is given by $\vec{r} = \vec{a} + \lambda\vec{v}$, where $\vec{a}$ is apposition vector on the line and $\vec{v}$ is some vector parallel to the line.

$$L: \quad \vec{r} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

b. M: $\dfrac{x-4}{1} = \dfrac{y-6}{2} = \dfrac{z-2}{3}$ gives $x = 4+\mu$, $y = 6+2\mu$ and $z = 2+3\mu$

Substitute into the equation of the plane P: $3x+4y-z=2$ we have

$$\begin{aligned} 3(4+\mu)+(6+2\mu)-(2+3\mu) &= 2 \\ 12+3\mu+24+8\mu-2-3\mu &= 2 \\ 34+8\mu &= 2 \\ 8\mu &= -32 \\ \mu &= -4 \end{aligned}$$

Substituting back gives $x = 4+(-4) = 0$, $y = 6+2(-4) = -2$ and

$z = 2+3(-4) = -10$

The point of intersection is (0, -2, -10).

c. $M: \quad \vec{r} = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix} + \mu \underbrace{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}_{\vec{v}}$ and $P: \quad \vec{r} \bullet \underbrace{\begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix}}_{\vec{n}} = 2$

$\vec{v} \bullet \vec{n} = (1)(3)+(2)(4)+(3)(-1) = 3+8-3 = 8$

$|\vec{v}| = \sqrt{1^2+2^2+3^2} = \sqrt{14} \quad |\vec{n}| = \sqrt{3^2+4^2+(-1)^2} = \sqrt{26}$

Let the required angle be θ where

$$\theta = \arcsin\left(\frac{\vec{v} \bullet \vec{n}}{|\vec{v}||\vec{n}|}\right) \arcsin\left(\frac{8}{\sqrt{14}\sqrt{26}}\right) = 24.79^\circ = 24.8^\circ$$

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d. $L: \quad \vec{r} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and $P: \quad \vec{r} \bullet \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} = 2$

$\vec{v}$ $\qquad$ $\vec{n}$

For line L and plane P to be parallel $\vec{v} \bullet \vec{n} = 0$

$\Rightarrow (2)(3) + (-1)(4) + (2)(-1) = 6 - 4 - 2 = 0$

Therefore the line L and plane P are parallel.

e. Embed the line L into a parallel plane to P, $\vec{r} \bullet \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} = D$

Taking a point on the line (4, -2, 1) we have

$\Rightarrow (4)(3) + (-2)(4) + (1)(-1) = 12 - 8 - 1 = 3$

We now find the distance between $\vec{r} \bullet \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} = 3$ and $\vec{r} \bullet \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} = 2$

Writing both in unit normal vector form $\vec{r} \bullet \hat{n} = d$, where d is the distance from the origin gives

$$\vec{r} \bullet \frac{1}{\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} = \frac{3}{\sqrt{26}} \text{ and } \vec{r} \bullet \frac{1}{\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ -1 \end{pmatrix} = \frac{2}{\sqrt{26}}$$

The required distance is $d = \frac{3-2}{\sqrt{26}} = \frac{1}{\sqrt{26}} = \frac{\sqrt{26}}{26}$

f. $L: \quad \vec{r} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and $M: \quad \vec{r} = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

For intersecting lines $4 + 2\lambda = 4 + \mu \quad \Rightarrow 2\lambda - \mu = 0$

$-2 - \lambda = 6 + 2\mu \quad \Rightarrow -\lambda - 2\mu = 8$

Solving simultaneously gives $\lambda = -\frac{8}{5}$ and $\mu = -\frac{16}{5}$

Now $z = 1 + 2\lambda = 1 + 2\left(-\frac{8}{5}\right) = -\frac{11}{5}$ and $z = 2 + 3\mu = 2 + 3\left(-\frac{16}{5}\right) = -\frac{38}{5}$.

Therefore the lines do not intersect.

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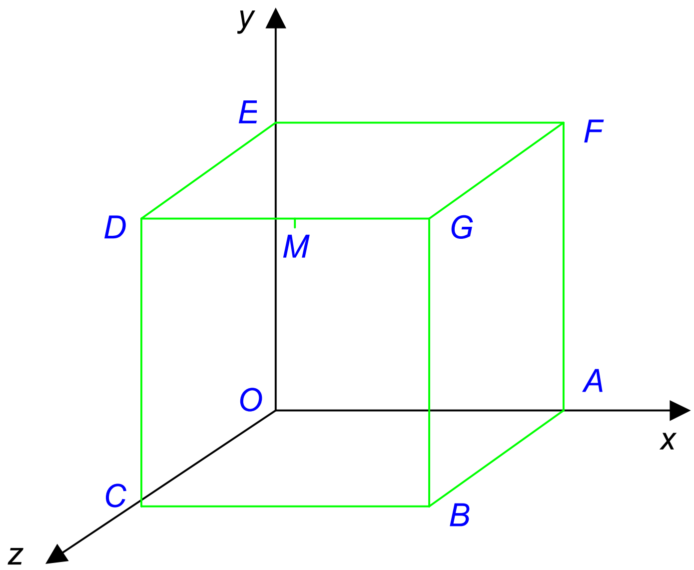
Question 2



OABCDEFG is a cube of side 6 units, *M* is the midpoint of *DG*.

a. Find the Cartesian equation of the plane *MAC*. Hence find the distance of from *G* to the plane *MAC*.

b. Find the vector equations of the lines *EB* and *MF*. Hence find the distance between the two lines.

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Solution to question 2

a. $\overrightarrow{OM}=\begin{pmatrix}3\\6\\6\end{pmatrix}, \overrightarrow{OA}=\begin{pmatrix}6\\0\\0\end{pmatrix}$ and $\overrightarrow{OC}=\begin{pmatrix}0\\0\\6\end{pmatrix}$

$$\overrightarrow{AM}=\overrightarrow{AO}+\overrightarrow{OM}=\begin{pmatrix}-6\\0\\0\end{pmatrix}+\begin{pmatrix}3\\6\\6\end{pmatrix}=\begin{pmatrix}-3\\6\\6\end{pmatrix}, \overrightarrow{AC}=\overrightarrow{AO}+\overrightarrow{OC}=\begin{pmatrix}-6\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\6\end{pmatrix}=\begin{pmatrix}-6\\0\\6\end{pmatrix}$$

$$\vec{r}=\overrightarrow{OA}+\lambda\overrightarrow{AM}+\mu\overrightarrow{AC}\Rightarrow\vec{r}=\begin{pmatrix}6\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}-3\\6\\6\end{pmatrix}+\mu\begin{pmatrix}-6\\0\\6\end{pmatrix}$$

$$\vec{n}=\overrightarrow{AM}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\-3&6&6\\-6&0&6\end{vmatrix}=(36-0)\vec{i}-(-18+36)\vec{j}+(0+36)\vec{k}$$

$$=36\vec{i}-18\vec{j}+36\vec{k}$$

$$=2\vec{i}-\vec{j}+2\vec{k}$$

Now $\vec{r}\bullet\left(2\vec{i}-\vec{j}+2\vec{k}\right)=D\Rightarrow 6(2)+0(-1)+0(2)=12$

$\Rightarrow\vec{r}\bullet\left(2\vec{i}-\vec{j}+2\vec{k}\right)=12\Rightarrow 2x-y+2z=12$

Embed point G into a parallel plane $G(6, 6, 6)$

$(6)(2)+(6)(-1)+(6)(2)=12-6+12=18\Rightarrow\vec{r}\bullet\left(2\vec{i}-\vec{j}+2\vec{k}\right)=18$

Now write in normal vector form $\vec{r}\bullet\hat{n}=d$ where d is the distance from the origin, we have

$\vec{r}\bullet\frac{1}{3}\left(2\vec{i}-\vec{j}+2\vec{k}\right)=\frac{12}{3}=4$ and $\vec{r}\bullet\frac{1}{3}\left(2\vec{i}-\vec{j}+2\vec{k}\right)=\frac{18}{3}=6$

The required distance is $=6-4=2$ units.

b. Line EB: $\overrightarrow{OE}=\begin{pmatrix}0\\6\\0\end{pmatrix}, \overrightarrow{OB}=\begin{pmatrix}6\\0\\6\end{pmatrix}, \overrightarrow{EB}=\overrightarrow{EO}+\overrightarrow{OB}=\begin{pmatrix}0\\-6\\0\end{pmatrix}+\begin{pmatrix}6\\0\\6\end{pmatrix}=\begin{pmatrix}6\\-6\\6\end{pmatrix}$

$\Rightarrow\vec{r}=\begin{pmatrix}0\\6\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$ there are other possibilities

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Line *MF*: $\overrightarrow{OM}=\begin{pmatrix}3\\6\\6\end{pmatrix}, \overrightarrow{OF}=\begin{pmatrix}6\\6\\0\end{pmatrix}, \overrightarrow{MF}=\overrightarrow{MO}+\overrightarrow{OF}=\begin{pmatrix}-3\\-6\\-6\end{pmatrix}+\begin{pmatrix}6\\6\\0\end{pmatrix}=\begin{pmatrix}3\\0\\-6\end{pmatrix}$

$\Rightarrow \vec{r}=\begin{pmatrix}3\\6\\6\end{pmatrix}+\mu\begin{pmatrix}1\\0\\-2\end{pmatrix}$ there are other possibilities

A common perpendicular :

$$\vec{a}=\vec{v}_1\times\vec{v}_2=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -1 & 1\\ 1 & 0 & -2\end{vmatrix}=(2-0)\vec{i}-(-2-1)\vec{j}+(0+1)\vec{k}$$

$$=2\vec{i}+3\vec{j}+\vec{k}$$

A vector joining both lines:

$$\vec{b}=-6\vec{j}+3\vec{i}+6\vec{j}+6\vec{k}=3\vec{i}+6\vec{k}$$

Using scalar projection of $\vec{b}$ onto $\vec{a}$ we have:

$$d=\left|\frac{\vec{a}\bullet\vec{b}}{|\vec{a}|}\right|=\left|\frac{(2)(3)+(3)(0)+(1)(6)}{\sqrt{2^2+3^2+1^2}}\right|=\left|\frac{6+0+6}{\sqrt{14}}\right|=\frac{12}{\sqrt{14}}=\frac{12\sqrt{14}}{14}=\frac{6\sqrt{14}}{7}.$$

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Question 3

3. Given the equations

$$2x - 3y + z = 5$$
$$x + y - 2z = 3$$
$$3x - 2y + az = b$$

a. Find the value of *a* for which the equations have no unique solution.

b. With *a* taking this value find the value *b* must take for there to be an infinite number of solutions to the equations. Interpret this situation geometrically giving any relevant equations in Cartesian form.

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Solution to question 3

a. First write the equations into matrix form:

$$\begin{aligned} 2x-3y+z&=5 \\ x+y-2z&=3 \\ 3x-2y+az&=b \end{aligned} \Rightarrow \begin{pmatrix} 2 & -3 & 1 \\ 1 & 1 & -2 \\ 3 & -2 & a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 6 \\ 3 \\ b \end{pmatrix}$$

Let $A=\begin{pmatrix} 2 & -3 & 1 \\ 1 & 1 & -2 \\ 3 & -2 & a \end{pmatrix}$. For a non-unique solution $\det(A)=0$.

$$\begin{aligned} 2(a-4)+3(a+6)+(-2-3)&=0 \\ 2a-8+3a+18-5&=0 \\ 5a+5&=0 \\ a&=-1 \end{aligned}$$

b. Solving by row elimination:

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}\begin{pmatrix} 2 & -3 & 1 \\ 1 & 1 & -2 \\ 3 & -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 5 \\ 3 \\ b \end{pmatrix} \Rightarrow \begin{matrix} R_1 \\ 2R_2-R_1 \\ 2R_3-3R_1 \end{matrix}\begin{pmatrix} 2 & -3 & 1 \\ 0 & 5 & -5 \\ 0 & 5 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 5 \\ 1 \\ 2b-15 \end{pmatrix}$$

$$\Rightarrow \begin{matrix} R_1 \\ R_2 \\ R_3-R_2 \end{matrix}\begin{pmatrix} 2 & -3 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 5 \\ 1 \\ 2b-16 \end{pmatrix}$$

For an infinite number of solutions $0z=2b-16 \Rightarrow 2b-16=0 \Rightarrow b=8$

Now let $z=\lambda$ $\quad 5y+5z=1 \Rightarrow 5y+5\lambda=1 \Rightarrow 5y=1-5\lambda \Rightarrow y=\dfrac{1-5\lambda}{5}$

$$2x-3y+z=5 \Rightarrow 2x-\frac{1-5\lambda}{5}+\lambda=5 \Rightarrow 2x=5+\frac{1-5\lambda}{5}-\lambda$$

$$\Rightarrow 2x=\frac{25+1-5\lambda-5\lambda}{5} \Rightarrow 2x=\frac{26-10\lambda}{5} \Rightarrow x=\frac{26-10\lambda}{10} \Rightarrow x=\frac{13-5\lambda}{5}$$

The Cartesian equation is $\dfrac{5x-13}{-5}=\dfrac{5y-1}{-5}=z$. The three planes meet on a common line.

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