Maths IB$_1$ HL

Test 5

Subjects : *Scalar product & equations of Planes*

Total : /

Thursday 27.02.2025

Name:

Question 1 (May 23) *[9 marks]*

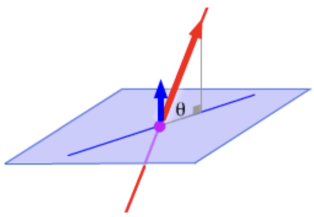
The angle between a line and a plane is α, where $\alpha \in \mathbb{R}$, $0 < \alpha < \dfrac{\pi}{2}$.

The equation of the line is $\dfrac{x-1}{3} = \dfrac{y+2}{2} = 5 - z$, and the equation of the plane is $4x + (\cos\alpha)y + (\sin\alpha)z = 1$.

i) Find a trigonometric equation for α (in other terms: α should be a solution of this equation)

Answers : $\vec{v} \cdot \vec{n} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} = 12 + 2\cos(\alpha) - \sin(\alpha)$

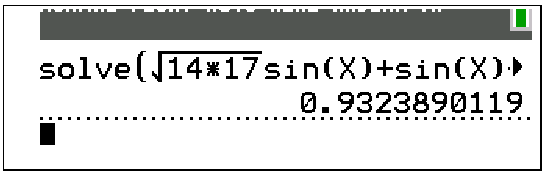
$\|\vec{v}\| = \sqrt{14}$ and $\|\vec{n}\| = \sqrt{17}$



then: $\cos\left(\frac{\pi}{2} - \alpha\right) = \dfrac{12 + 2\cos(\alpha) - \sin(\alpha)}{\sqrt{14}\sqrt{17}}$

ii) To solve this equation, we have to use the function *SOLVE* of a calculator.

The result should be $\alpha = 54.4219...^0 \;\; = 0.932389...$rad



iii) If we take $\alpha = \frac{3\pi}{2}$, the equation of the line becomes : $4x + 0y + -z - 1 = 0$

Then we substitute :$x = 1 + 3\lambda$, $y = -2 + 2\lambda$, $z = 5 - \lambda$

So $4(1 + 3\lambda) + 0 - 1(5 - \lambda) - 1 = 0$

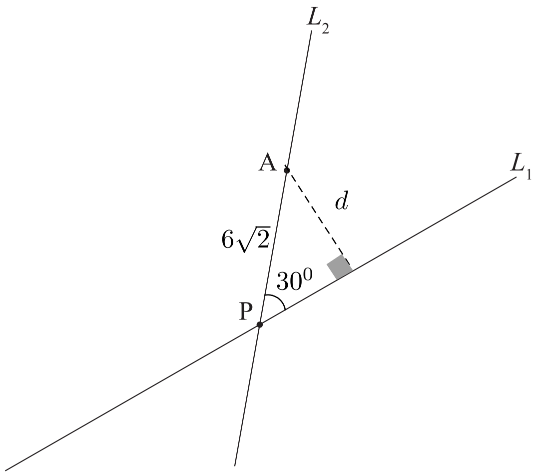
$4 + 12\lambda - 5 + \lambda - 1 = 0 \quad \Rightarrow \lambda = \frac{2}{13}$ and then $x = 1 + 3\left(\frac{2}{13}\right)$, $y = -2 + 2\left(\frac{2}{13}\right)$, $z = 5 - \left(\frac{2}{13}\right)$

$\Rightarrow \quad x = \frac{19}{13},\; y = -\frac{22}{13},\; z = \frac{63}{13}$ $\quad \boxed{\mathrm{I} : \left(\frac{19}{13}, -\frac{22}{13}, \frac{63}{13}\right)}$

Question 2 *[6 marks]*

Two lines, L_1 and L_2, intersect at point P. Point $\mathrm{A}(2t, 8, 3)$, where $t > 0$, lies on L_2. This is shown in the following diagram.

diagram not to scale



The acute angle between the two lines is $\frac{\pi}{3}$.

The direction vector of L_1 is $\begin{pmatrix}1\\1\\0\end{pmatrix}$, and $\vec{\mathrm{PA}} = \begin{pmatrix}2t\\0\\3+t\end{pmatrix}$.

(a) Show that $4t = \sqrt{10t^2 + 12t + 18}$. [4]

(b) Find the value of t. [4]

(c) Hence or otherwise, find the shortest distance from A to L_1. [4]

(d) Find a point B that lies on L_1 [2]

(e) Find a vector equation of the plane Π_{APB} that contains A B an C [4]

(f) Find a cartesian equation of the plane Π_{ABP} [3]

(g) From (f) otherwise, find a vector $\vec{n}$ *perpendicular* to L_1 and L_2 [3]

a) Using the dot product : $\cos\left(\frac{\pi}{3}\right) = \frac{2t}{\sqrt{2}\sqrt{4t^2+(3+t)^2}} \Rightarrow 2t = \frac{1}{2}\sqrt{2}\sqrt{4t^2+t^2+6t+9}$

$\Rightarrow 4t = \sqrt{10t^2+12t+18}$

b) $\Rightarrow 16t^2 = 10t^2 + 12t + 18 \Rightarrow 6t^2 - 12t - 18 = 0 \Rightarrow t^2 - 2t - 3 = 0 \quad \Delta = 16 \quad t = \frac{2 \pm 4}{2} = \boxed{3s}\,(>0)$

c) Then we can find the vector $\overrightarrow{\mathrm{PA}}$: $\overrightarrow{\mathrm{PA}} = \begin{pmatrix}6\\0\\6\end{pmatrix}$ and $\|\overrightarrow{\mathrm{PA}}\| = 6\sqrt{2}$ then $d = 6\sqrt{2}\sin\left(\frac{\pi}{3}\right) = \boxed{3\sqrt{6}}$

d) To find a point B that lies on L_1 we start from A: $(6, 8, 3)$

Then we find P, because $\overrightarrow{PA} = \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix}$ and $\overrightarrow{PA} = \overrightarrow{OP} - \overrightarrow{OA}$

$$\Rightarrow \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{PA} = \begin{pmatrix} 6 \\ 8 \\ 3 \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \\ 9 \end{pmatrix} \quad P: (12, 8, 9)$$

Then $\overrightarrow{OB} = \overrightarrow{OP} + \lambda \vec{v}$ where $\vec{v}$ is the given director vector $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

We can chose any real value for λ, foe example $\lambda, = 1$ gives $\begin{pmatrix} x_B \\ y_B \\ z_B \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \\ 9 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

gives B:$(13, 9, 9)$

e) A vector equation of Π_{ABC} is $\overrightarrow{OM} = \overrightarrow{OP} + \lambda_1 \overrightarrow{PA} + \lambda_2 \overrightarrow{PB}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 8 \\ 9 \end{pmatrix} + \lambda_1 \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

f) To get a cartesian equation :

$$\begin{cases} x = 12 + 6\lambda_1 + 1\lambda_2 \\ y = 8 \quad\quad\quad + 1\lambda_2 \\ z = 9 + 6\lambda_1 + 0\lambda_2 \end{cases}$$

$x - y = 4 + 6\lambda_1$

$z = 9 + 6\lambda_1$

$\Rightarrow (x - y) - z = -5 \Rightarrow \boxed{x - y - z + 5 = 0}$

(g) From (f) we know a *vector* $\vec{n}$ *normal* to Π_{ABC}: $\vec{n} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

As L_1 and L_2 are on Π_{ABC}, this vector $\vec{n}$

We can verify it using the director vectors of L_1 and L_2 :

$$\vec{n} \perp L_1 \Leftrightarrow \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 0 \\ 6 \end{pmatrix} = 0 \quad \text{and} \quad \vec{n} \perp L_2 \Leftrightarrow \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Question 3 *[4 marks]*

A plane, which is parallel to the plane $x - 2y + 3z = 2$, passes through the points $(-1, -1, 1)$ and $(-1, 2, k)$. Find the value of k.

$-1 + 2 + 3 = 2$ ok

$-1 - 4 + 3k = 2 \quad \Rightarrow \boxed{k = \frac{7}{3}}$