Maths IB_1 HL

Test 2

Subjects : *Complex numbers #1...*

Total : / 18

Thuersday 7.11.2024

Answers

Problem 1

[6 marks]

Consider the complex numbers $z_1 = 1 + \sqrt{3}\,i$, $z_2 = 1 + i$ and $w = \dfrac{z_1}{z_2}$.

(a) By expressing z_1 and z_2 in modulus-argument form write down

(i) the modulus of w; $\rho = |w| = \sqrt{1^2 + (\sqrt{3})^2} = \boxed{2}$

(ii) the argument of w. $\theta = \arctan\left(\frac{\text{Im}(w)}{\text{Re}(w)}\right) = \arctan\left((\sqrt{3}\right) = \boxed{\frac{\pi}{3}}$ [4]

(b) Find the smallest positive integer value of n, such that w^n is a real number. [2]

$w^n = \rho^n \text{cis}(n\theta) = 2^n \text{cis}\left(n\frac{\pi}{3}\right)$ is real for $n\frac{\pi}{3} = \pi$ then for $\boxed{\text{n}=3}$ $(w^2 = -8)$

Problem 2

[6 marks]

It is given that $z = 5 + qi$ satisfies the equation $z^2 + iz = -p + 25i$, where $p, q \in \mathbb{R}$.

Find the value of p and the value of q.

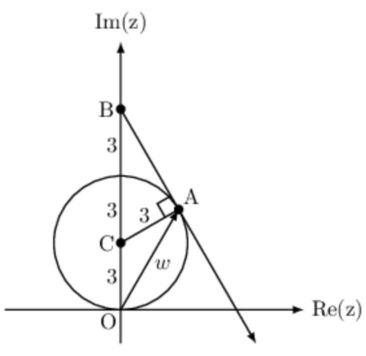
$(5 + iq)^2 + i(5 + iq) = -p + 25i$

$25 + 10iq - q^2 + 5i - q = -p + 25i \quad \Rightarrow \begin{cases} 25 - q^2 - q = -p \\ 10q + 5 = 25 \end{cases} \quad \Rightarrow \boxed{q = 2} \quad \Rightarrow p = q^2 + q - 25 = \boxed{-19}$

Problem 3

[6 marks]

(a) Given that the angle between a tangent and a radius is a right angle, we have



Using the right triangle ABC, we find

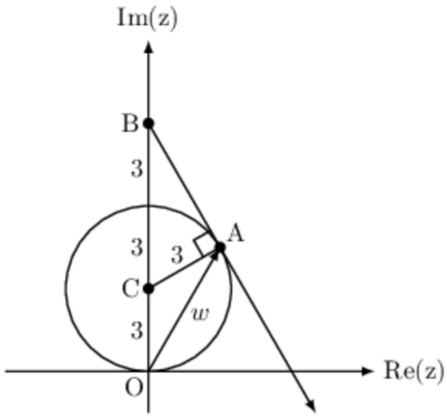
$$\cos(\text{A}\hat{\text{C}}\text{B}) = \frac{3}{6} \qquad \left(= \frac{1}{2}\right)$$
$$\text{A}\hat{\text{C}}\text{B} = \frac{\pi}{3}$$
$$\text{A}\hat{\text{C}}\text{O} = \frac{2\pi}{3}$$

Hence, using the cosine rule, we get

$$|w|^2 = 3^2 + 3^2 - 2(3)(3)\cos\left(\frac{2\pi}{3}\right)$$
$$|w|^2 = 18 - 18\left[-\frac{1}{2}\right]$$
$$|w|^2 = 27$$

$\Rightarrow \boxed{|\omega| = \sqrt{27}}$

problem 3 continuing …



(b) Using the isosceles triangle OAC, we have

$$\begin{aligned} \text{A}\hat{\text{O}}\text{C} &= \frac{1}{2}(\pi - \text{A}\hat{\text{C}}\text{O}) \\ &= \frac{1}{2}\left[\pi - \frac{2\pi}{3}\right] \\ &= \frac{\pi}{6} \end{aligned}$$

Hence the argument of w is

$$\begin{aligned} \arg w &= \frac{\pi}{2} - \text{A}\hat{\text{O}}\text{C} \\ &= \frac{\pi}{2} - \frac{\pi}{6} \end{aligned}$$

$$\Rightarrow \boxed{\phi = \frac{\pi}{3}}$$

(c) If we write w in Cartesian form, we obtain

$$\begin{aligned} w &= 3\sqrt{3}\,\text{cis}\left(\frac{\pi}{3}\right) \\ &= 3\sqrt{3}\left[\cos\left(\frac{\pi}{3}\right) + \text{i}\sin\left(\frac{\pi}{3}\right)\right] \\ &= 3\sqrt{3}\left[\frac{1}{2} + \text{i}\frac{\sqrt{3}}{2}\right] \end{aligned}$$

$$\Rightarrow \boxed{\omega = \frac{3\sqrt{3}}{2} + \frac{9}{2}i}$$