Christmas Examination

Wednesday 11 Dec.2024

Maths SL IB$_2$

Part 2

(7 Problems 55 marks)

ANSWERS

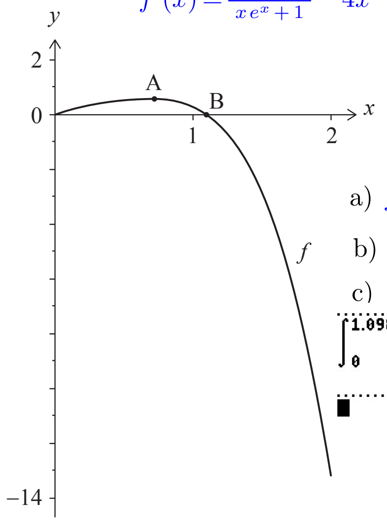
A calculator **is allowed** for this second part

Problem 1

[/ 6 marks]

The function f is defined as $f(x) = \ln(xe^x + 1) - x^4$, for $0 \le x \le 2$. The graph of f is shown in the following diagram.

$f'(x) = \frac{e^x + x e^x}{x e^x + 1} \quad 4x^3$



a) $f'(x) = 0$ for $x \cong 0.708$

b) $f(x) = 0$ for $x \cong 1.09885$

c)

$\int_0^{1.09855} (\ln(e^x+1)-x^4)dx$

0.7968373939

The graph of f has a local maximum at point A. The graph intersects the x-axis at the origin and at point B.

(a) Find the coordinates of A. [2]

(b) Find the x-coordinate of B. [1]

(c) Find the total area enclosed by the graph of f, the x-axis and the line $x = 2$. [3]

Problem 2

[/ 2 marks]

A particle moves along a straight line. Its displacement, s metres, from a fixed point O after time t seconds is given by $s(t) = 4.3\sin\left(\sqrt{3t+5}\right)$, where $0 \le t \le 10$.

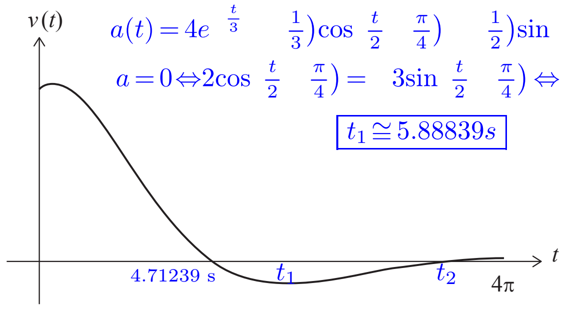
The particle first comes to rest after q seconds.

(a) Find the value of q. $v(t) = \frac{4.3\cos(\sqrt{3t+5})\cdot 3}{2\sqrt{3t+1}}$ $\boxed{v(t) = 0 \quad \text{for } t = q = 18.895\,s}$ [2]

Problem 3

[/ 4 marks]

A particle moves in a straight line such that its velocity, $v\,\mathrm{m\,s}^{-1}$, at time t seconds is given by $v(t) = 4e^{-\frac{t}{3}}\cos\left(\frac{t}{2} - \frac{\pi}{4}\right)$, for $0 \le t \le 4\pi$. The graph of v is shown in the following diagram.



$a(t) = 4e^{\frac{t}{3}}\ \ \frac{1}{3})\cos\ \frac{t}{2}\ \ \frac{\pi}{4})\ \ \ \frac{1}{2})\sin\ \frac{t}{2}\ \ \frac{\pi}{4}))$

$a = 0 \Leftrightarrow 2\cos\ \frac{t}{2}\ \ \frac{\pi}{4}) =\ \ 3\sin\ \frac{t}{2}\ \ \frac{\pi}{4}) \Leftrightarrow \tan\ \frac{t}{2}\ \ \frac{\pi}{4}) =\ \ \frac{3}{2}$

$\boxed{t_1 \cong 5.88839 s}$

Let t_1 be the first time when the particle's **acceleration** is zero.

(a) Find the value of t_1. [2]

Let t_2 be the **second** time when the particle is instantaneously at rest.

(b) Find the value of t_2. $\boxed{t_2 = 10.9956\ \mathrm{s}}$ [2]

Problem 4

[/ 16 marks]

Consider a function f. The line L_1 with equation $y = 3x + 1$ is a tangent to the graph of f when $x = 2$.

(a) (i) Write down $f'(2)$. $\boxed{f'(2) = 3}$ (as 3 is the gradient of the tangent at $x = 2$)

(ii) Find $f(2)$. $\boxed{f(2) = 7}$ [4]

Let $g(x) = f(x^2 + 1)$ and P be the point on the graph of g where $x = 1$.

(b) Show that the graph of g has a gradient of 6 at P. At P, $g'(x) = f'(x^2+1) \times 2x = 6$ [5]

(c) Let L_2 be the tangent to the graph of g at P. L_1 intersects L_2 at the point Q. Find the y-coordinate of Q. [7]

L_1: $y = 3x + 1$

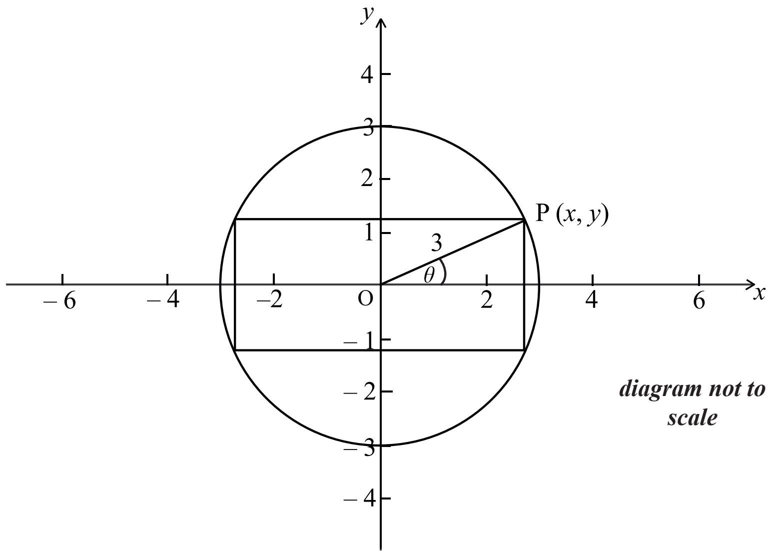
L_2: $y = 6x + 1$

intersection at $\boxed{(0,1)}$

Problem 5

[/ 13 marks]

A rectangle is inscribed in a circle of radius 3 cm and centre O, as shown below.



The point $P(x, y)$ is a vertex of the rectangle and also lies on the circle. The angle between (OP) and the x-axis is θ radians, where $0 \le \theta \le \frac{\pi}{2}$.

(a) Write down an expression in terms of θ for

(i) x; $x = 3\cos(\theta)$

(ii) y. $y = 3\sin(\theta)$ *[2 marks]*

Let the area of the rectangle be A.

(b) Show that $A = 18\sin 2\theta$. $A = xy = 9\cos(\theta)\sin(\theta) = 18\sin(2\theta)$ *[3 marks]*

(c) (i) Find $\frac{dA}{d\theta}$. $\boxed{\frac{dA}{d\theta} = 36\cos(2\theta)}$ and $\frac{d^2A}{d\theta^2} = \ 72\sin(2\theta)$

(ii) Hence, find the exact value of θ which maximizes the area of the rectangle. $\cos(2\theta) = 0$ for $2\theta = \frac{\pi}{2}$ $\Rightarrow$ $\boxed{\theta = \frac{\pi}{4}}$

(iii) Use the second derivative to justify that this value of θ does give a maximum. $\frac{d^2A}{d\theta^2}\ \frac{\pi}{4}) = \ 72\sin\ \frac{\pi}{2}) < 0$ $\Rightarrow$ *max !* *[8 marks]*

Problem 6 [/ 8 marks]

Notice $(f \circ g)(x)$ is a notation for $f(g(x))$

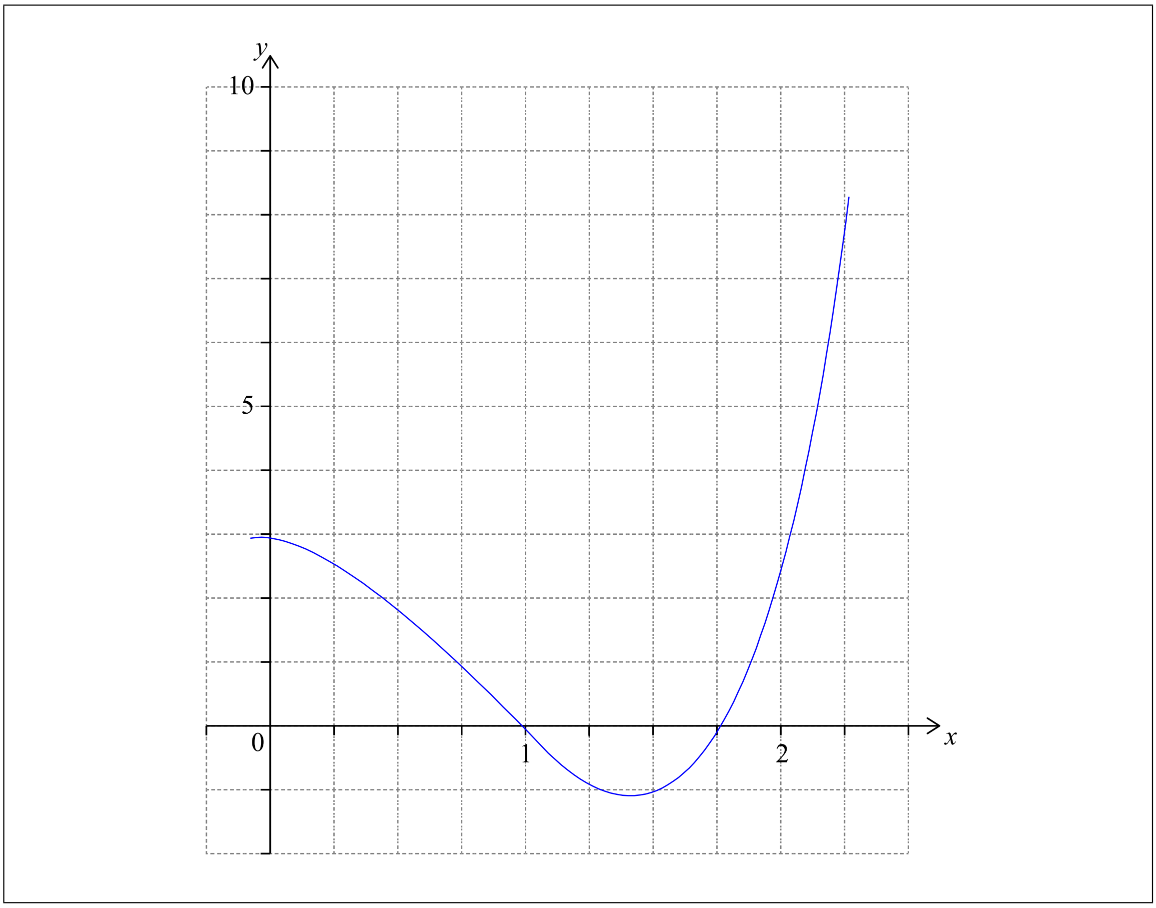
Let $f(x) = x^2 - 1$ and $g(x) = x^2 - 2$, for $x \in \mathbb{R}$.

a) $f(g(x)) = (x^2 \quad 2)^2 \quad 1 = x^3 \quad 4x^2 + 3$

b) see the curve

(a) Show that $(f \circ g)(x) = x^4 - 4x^2 + 3$. [2]

(b) On the following grid, sketch the graph of $(f \circ g)(x)$, for $0 \le x \le 2.25$. [3]

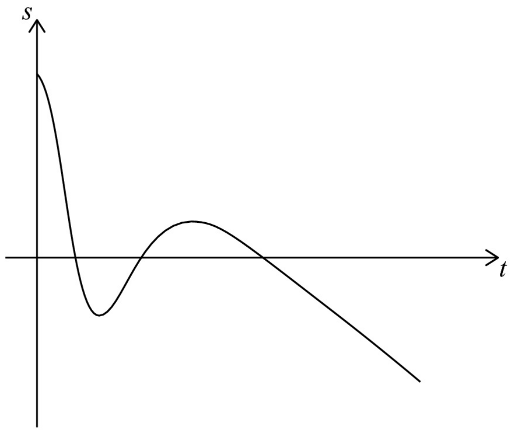


(c) The equation $(f \circ g)(x) = k$ has exactly two solutions, for $0 \le x \le 2.25$. Find the possible values of k. [3]

Problem 7

[/ 6 marks]

Particle A is moving along a straight line such that its displacement from a point P, after t seconds, is given by $s_A = 15 - t - 6t^3 e^{-0.8t}$, $0 \le t \le 25$. This is shown in the following diagram.



(a) Find the initial displacement of particle A from point P. $s(0) = 15m$ [2]

(b) Find the value of t when particle A first reaches point P. $s = 0 \text{ for } t = 6.79321s$ [2]

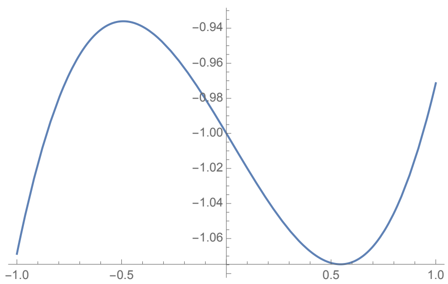
(c) Find the value of t when particle A first changes direction. $v(t) = \ 1 \quad 18t^2 e^{\ 0.8t} + 4.8t^3 e^{\ 0.8t}$ [2]

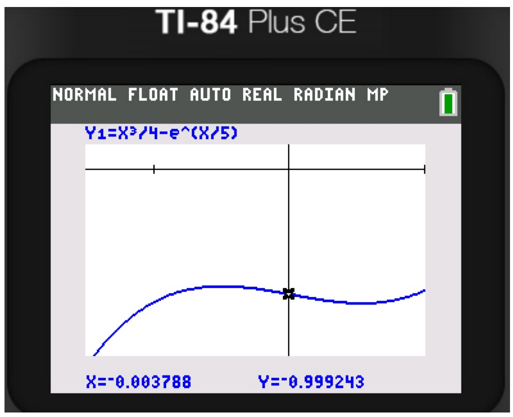
$\boxed{v(t) = 0 \text{ at } t = 10.0145s}$

Bonus

[+ 5]

The figure below shows the graph of $f(x) = \frac{x^3}{4} \quad e^{\frac{x}{5}}$, for $\ 1 < x < 1$





Using your calculator, we find the x of the inflexion point, with four significant digits.