Christmas Examination

Wednesay 11 Dec. 2024

Maths SL IB_2

Part 1

(7 Problems 93 marks)

ANSWERS

A calculator is not allowed for this fisrt part

Problem 1

[/ 5 marks]

We consider the function $f(x) = 2\cos(x)e^x$

a) The derivative of $f(x)$ is : $\boxed{f'(x) = 2e^x(\cos(x) - \sin(x))}$

b) $f'(0) = 2e^0(\cos(0) - \sin(0)) = \boxed{2}$

c) The graph of f has an *horizontal* tangent line $\Leftrightarrow \cos(x) - \sin(x) \Leftrightarrow \cos(x) = \sin(x) \Leftrightarrow \tan(x) = 1$

that is the case for $\boxed{x_0 = \frac{\pi}{4}}$ $\quad \left(0 < x_0 < \frac{\pi}{2}\right)$

Problem 2

[/ 16 marks]

$f(x) = (x-1)^2 e^{2x}$

1) A value of x such that the tangent to the curve of equation $y = f(x)$ is *horizontal*

is a solution of $f'(x) = 0$

with $f'(x) = 2(x-1)e^{2x} + (x-1)^2\, 2e^{2x} = 2e^{2x}(x-1)(1+x-1) = 2e^{2x}x\,(x-1)$

then $\boxed{x=0}$ or $\boxed{x=1}$

2) $f''(x) = 2[e^{2x}(x^2-x)]' = 2[2e^{2x}(x^2-x) + e^{2x}(2x-1)] = 2e^{2x}[2(x^2-x) + (2x-1)] = 2e^{2x}(2x^2-1)$

3) $f''(0) = -2 < 0 \Rightarrow \boxed{\textit{maximum at } x = 0}$

$f''(1) = 2e^2(2-1) = 2e^2 > 0 \Rightarrow \boxed{\textit{minimum at } x = 1}$

Problem 3 [/ 16 marks]

(a) (i) $f'(2)$ is the gradient of the tangent to the curve at $x=2$.

As this tangent L_1 has equation: $y=3x+1$, then $\boxed{f'(2)=3}$

(ii) $f(2)$ is equal to $y=3\times 2+1=\boxed{7}$

$g(x)=f(x^2+1)$ then where $x=1$, $g(x)=g(1)=f(2)=7$

(b) $g'(x)=f'(x^2+1)*2x$ then $g'(1)=f'(2)\times 2=6$

(c) L_2 is the tangent to the graph of g where $x=1$

L_2 has equation $y=6x+b$ and as we saw, $y=7$ for $x=1$, then $7=6+b \Rightarrow b=1$

Then L_2 has equation : $y=6x+1$

Q=$L_1\cap L_2$, then the $x-$coordinates of Q is solution of $3x+1=6x+1 \Rightarrow x=0$ and $y=1$

$\boxed{Q:(0,1)}$

Problem 4 [/ 9 marks]

$v(t)=72(1-e^{-0.16t})$ where v is in m/s and t in seconds

a) Find the velocity of the ball at [2]

i) $t=0$ is $72(1-1)=\boxed{0ms^{-1}}$

ii) $t=8$ sec $72(1-e^{-1.28})=\boxed{51.98ms^{-1}}$

b) [6]

i) $a(t)=72(0.16e^{-0.16t})=\boxed{11.54e^{-0.16t}}$

ii) $\boxed{a(0)=11.54\ ms^{-2}}$

c) [6

i) As t becomes *large*, $\boxed{v(t)\to 72ms^{-1}}$

ii) As t becomes *large*, $\boxed{a(t)\to 72ms^{-1}}$

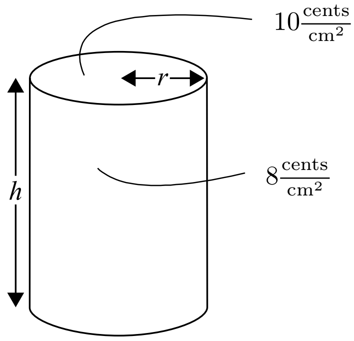
iii) A the speed reach asymptotically its maximum value (and then varies less and less)

the acceleration diminues and becomes zero.

Problem 5 [/ 15 marks]

A closed cylindrical can with radius r centimetres and height h centimetres has a volume of 20π cm^3.

diagram not to scale



(a) $V = \pi r^2 h$ then $\pi r^2 h = 20\pi$

$\Rightarrow \boxed{h = \frac{20}{r^2}}$

(b) Cost$=2\pi r^2 \times 10 + 2\pi r h \times 8$

$=(20\pi r^2 + 16\pi r \frac{20}{r^2}$

$= 20\pi r^2 + \frac{320\pi}{r}$

(c) the value or r for C min

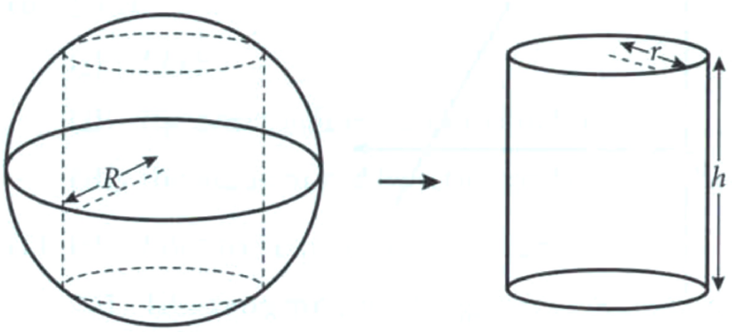
is solution of $\frac{dC}{dr} = 0 \Rightarrow 40\pi r - \frac{320\pi}{r^2} = 0$

$\Rightarrow r^r = 8 \quad \Rightarrow \boxed{r = 2}$

Hence the minimal cost is $C(r=2) = \boxed{240\pi \text{ cents}}$

Problem 6 [/ 13 marks]

Find the hight h and the base radius r of the largest right circular cylinder that can be made by cutting it away from a sphere with a radius of R.



$r = R\cos(\theta) \quad h = 2R\sin(\theta) \quad v(\theta) = \pi r^2 h = 2\pi R^2 \cos^2(\theta)\sin(\theta)$

$\frac{dv}{d\theta} = 8\pi[\cos^2(\theta)\cos(\theta) + 2\cos(\theta)\sin^2(\theta)] = 2\pi R^3 \cos(\theta)[\cos^2(\theta) + 2\sin^2(\theta)]$

$2\pi R^3 \cos(\theta)[1 - 3\sin^2(\theta)]$

$\frac{dv}{d\theta} = 0$ for $\cos(\theta) = 0$ and for $1 - 3\sin^2(\theta) = 0$

that is: for $\theta = \frac{\pi}{2}$ of $\sin(\theta) = \sqrt{\frac{1}{3}}$

$\theta = \frac{\pi}{2}$ gives a minimum hence the solution is $\theta = \arcsin\left(\sqrt{\frac{1}{3}}\right) \cong \boxed{35.26^0}$

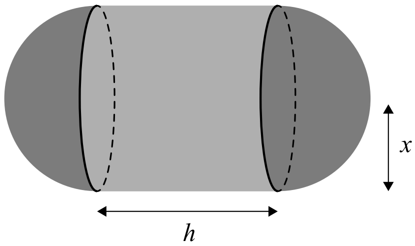
therefore : the radius is $\boxed{r = \mathrm{R}\cos(35.26^0) \cong 0.577}$

the hight is $\boxed{h = 2\,\mathrm{R}\sin(35.26^0) = 1.15}$

And the volume itself is $\pi r^2 h = 1.2R^3$

Problem 7 [/ 14 marks]

The solid shown in the following diagram is comprised of a cylinder and two hemispheres. The cylinder has height h cm and radius x cm. The hemispheres fit exactly onto either end of the cylinder.



The volume of the cylinder is $41\,\mathrm{cm}^3$.

(a) Show that the total surface area, $S\,\mathrm{cm}^2$, of the solid is given by $S = \dfrac{82}{x} + 4\pi x^2$. [3]

The total surface area of the solid has a local maximum or a local minimum value when $x = a$.

(b) (i) Find an expression for $\dfrac{\mathrm{d}S}{\mathrm{d}x}$.

(ii) Hence, find the **exact** value of a. [5]

(c) (i) Find an expression for $\dfrac{\mathrm{d}^2S}{\mathrm{d}x^2}$.

(ii) Use the second derivative of S to justify that S is a minimum when $x = a$.

(iii) Find the minimum surface area of the solid. [6]

(a) That is : $S = 2\pi r h + 2\times 2\pi r^2 = 2\pi r(h + 2r)$ with $V_{\text{cylindre}} = \pi r^2 h = 41\mathrm{cm}^3$

then from the second equation : $h = \frac{41}{\pi r^2}$

$$\Rightarrow S = 2\pi r\left(\frac{41}{\pi r^2} + 2r\right) = \left(\frac{82}{r} + 4\pi r^2\right) = \frac{82}{r} + 4\pi r^2 \quad \Rightarrow S(x) = \frac{82}{x} + 4\pi x^2 \qquad (x = r)$$

(b) i) $\frac{dS}{dx} = -\frac{82}{x^2} + 8\pi x \qquad \Rightarrow \frac{dS}{dx} = 0$ for $x^3 = \frac{82}{8\pi} \qquad \Rightarrow$ (ii): $\boxed{a = \sqrt[3]{\frac{82}{8\pi}} \cong 1.48}$

(c) i) $\frac{d^2S}{dx^2} = -2\frac{82}{x^3} + 8\pi \qquad \Rightarrow \frac{d^2S}{dx^2}(a) = \frac{-164}{\left(\sqrt[3]{\frac{82}{8\pi}}\right)^3} + 8\pi = \frac{-164}{\frac{82}{8\pi}} + 8\pi = \frac{-164\times 8\pi}{82} + 8\pi = -2 + 8\pi\left(\frac{1}{82} + 1\right) > 0$

(ii) then a gives a *minimum* of S

(iii) This minimum surface is $\boxed{S(a) = \frac{82}{a} + 4\pi a^2 = \frac{82}{\sqrt[3]{\frac{82}{8\pi}}} + 4\pi\left(\sqrt[3]{\frac{82}{8\pi}}\right)^2 \cong 83\mathrm{cm}^3}$