Christmas Examination Wednesay 13 December 2023

Maths SL IB_2

Part 1

(7 Problems 64 marks)

ANSWERS

Problem 1 [/ 8 marks]

1) The function f is define by $f(x) = \frac{x^2}{e^x}$.

i) $f'(x) = \frac{(x^2)' e^x - x^2 (e^x)'}{(e^x)^2} = \frac{x(2-x)}{e^x}$ [3]

ii) The gradient to the curve $y = f(x)$ is equal to $\frac{1}{e}$ for $\boxed{x=1}$ [2]

2) The function g is define by $g(x) = e^{x^2+1}$

Then $g'(x) = 2x e^{x^2+1}$ and $\boxed{g'(-1) = -2e^2}$ [3]

Problem 2 [/ 6 marks]

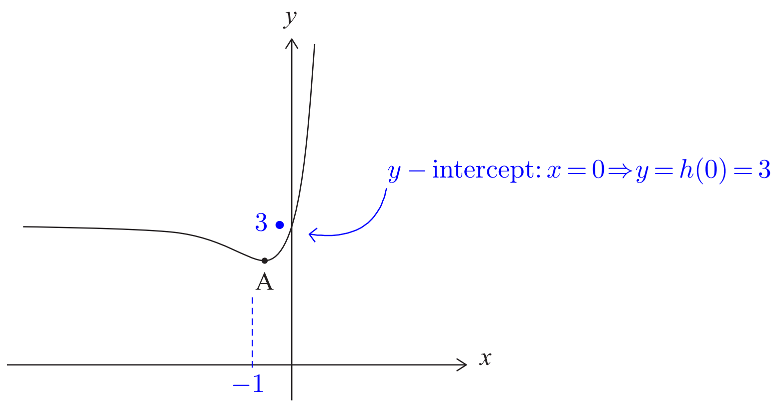
The derivative of the function f is given by $f'(x) = \frac{6x}{x^2+1}$.

The graph of $y = f(x)$ passes through the point $(1, 5)$. Find an expression for $f(x)$.

$f(x) = \int \frac{6x}{x^2+1} dx = 3\ln(x^2+1) + c$ as $f(1) = 5$ then $\boxed{c = 5 - 3\ln(2)}$

Problem 3 [/ 13 marks]

The function h is defined by $h(x) = 2xe^x + 3$, for $x \in \mathbb{R}$. The following diagram shows part of the graph of h, which has a local minimum at point A.



(a) Find the value of the y-intercept. $\boxed{y = 3}$ [2]

(b) Find $h'(x)$. $\boxed{h'(x) = 2e^x(x+1)}$ [2]

(c) Hence, find the coordinates of A. $\boxed{x_A = -1}$ and $\boxed{y_{A=-\frac{2}{e}+3}}$ [5]

(d) (i) Show that $h''(x) = (2x+4)e^x$ $h''(x) = (h'(x))' = 2e^x(x+1) + 2e^x(1+0) = 2e^x(x+2)$

(ii) Find the values of x for which the graph of h is concave-up $\Leftrightarrow h''(x) > 0 \Leftrightarrow \boxed{x > -2}$ [4]

Problem 6 [/ 3 marks]

Events A and B are independent and $P(A) = 3P(B)$.

Given that $P(A \cup B) = 0.68$, find $P(B)$.

Here: $P(A \cup B) = P(A) + P(B) = 4P(B)$ then $\boxed{P(B) = \frac{0.68}{4} = 0.17}$

Problem 7 [/ 3 marks]

Events A and B are independent and P(A) = 3P(B).

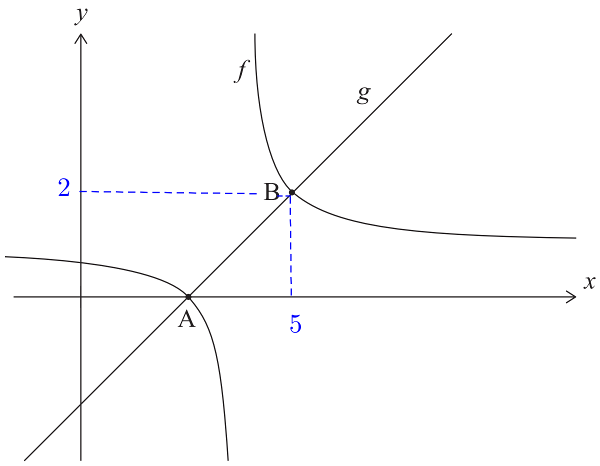
Given that P(A [B) = 0.77, find P(B).

Here: P(A|B)=P(A) then $\boxed{P(B) = \frac{0.77}{3} = 0.257}$

Problem 4 [/ 15 marks]

Consider the functions $f(x) = \dfrac{1}{x-4} + 1$, for $x \neq 4$, and $g(x) = x - 3$ for $x \in \mathbb{R}$.

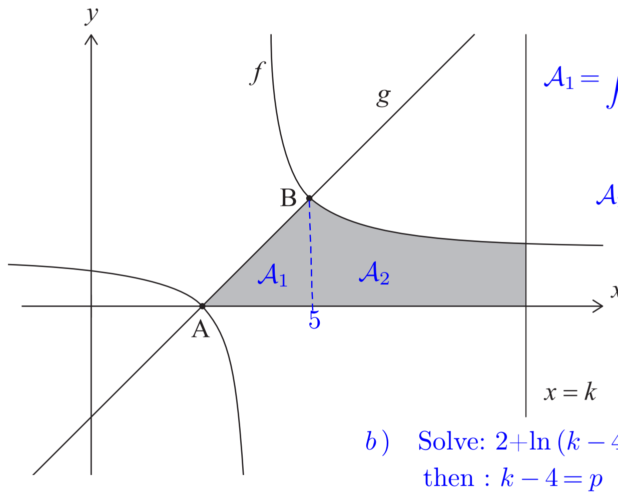
The following diagram shows the graphs of f and g.



The graphs of f and g intersect at points A and B. The coordinates of A are $(3, 0)$.

(a) Find the coordinates of B. Solving $f(x) = g(x)$: $(x-4)^2 = 1 \Rightarrow x = 4 \pm 1$ $\boxed{\text{B:}(5,2)}$ [5]

In the following diagram, the shaded region is enclosed by the graph of f, the graph of g, the x-axis, and the line $x = k$, where $k \in \mathbb{Z}$.



$\mathcal{A}_1 = \int_3^5 g(x)\,dx = \left[\frac{x^2}{2} - 3x\right]_3^5$

$= 2$

$\mathcal{A}_2 = \int_5^k f(x)\,dx$

$= [\ln(|x-4|) + x]_5^k$

$= \ln(k-4) + k - 5$

b) Solve: $2 + \ln(k-4) + k - 5 = \ln(p) + 8$

then : $k - 4 = p$ and $-3 + k = 8$

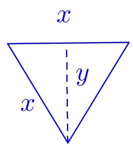
The area of the shaded region can be written as $\ln(p) + 8$, where $p \in \mathbb{Z}$.

(b) Find the value of k and the value of p. $\boxed{k = 11 \quad \text{and} \quad p = 11 - 4 = 7}$ [10]

Problem 5 [/ 16 marks]

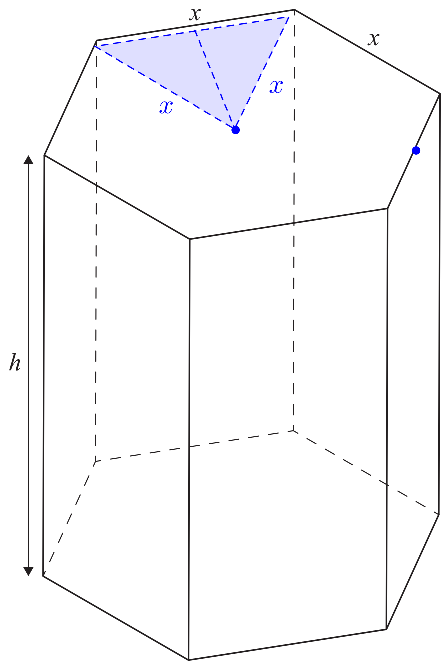
A hollow chocolate box is manufactured in the form of a right prism with a regular hexagonal base. The height of the prism is h cm, and the top and base of the prism have sides of length x cm.

diagram not to scale



$\frac{y}{x} = \sin(60)$

$y = x\frac{\sqrt{3}}{2}$



(a) Area $= 6\frac{xy}{2}$

$= 3x \times x\frac{\sqrt{3}}{2}$

(b) $1200 = 3\frac{\sqrt{3}}{2}x^2 + 6xh$

$400 = \frac{\sqrt{3}}{2}x^2 + 2xh$

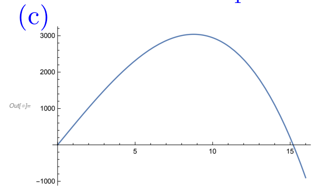
$\Rightarrow h = \frac{400 - \frac{\sqrt{3}}{2}x^2}{2x}$

$V = Ah$

$= 3\frac{\sqrt{3}}{2}x^2 \times \frac{400 - \frac{\sqrt{3}}{2}x^2}{2x}$

$= 300\sqrt{3}x - \frac{9}{4}x^3$

(c)



(a) Given that $\sin 60° = \frac{\sqrt{3}}{2}$, show that the area of the base of the box is equal to $\frac{3\sqrt{3}x^2}{2}$. [2]

(b) Given that the total external surface area of the box is $1200\,\text{cm}^2$, show that the volume of the box may be expressed as $V = 300\sqrt{3}x - \frac{9}{4}x^3$. [5]

(c) Sketch the graph of $V = 300\sqrt{3}x - \frac{9}{4}x^3$, for $0 \le x \le 16$. [2]

(d) Find an expression for $\frac{dV}{dx}$. $= 300\sqrt{3} - \frac{27}{4}x^2$ [2]

(e) Find the value of x which maximizes the volume of the box. $x_m = \sqrt{\frac{400\sqrt{3}}{9}} \cong 8.77$ [2]

(f) Hence, or otherwise, find the maximum possible volume of the box. $V_{\max} = V(x_m)$ [2]

The box will contain spherical chocolates. The production manager assumes that they can calculate the exact number of chocolates in each box by dividing the volume of the box by the volume of a single chocolate and then rounding down to the nearest integer.

(g) Explain why the production manager is incorrect. Empty space between chocolates … [1]

Problem 6 …

Problem 7 … see page 2

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