MATHS AA

June Exam

Total : / 138 marks

Friday 14 June 2024

Total duration : 3 hours

ANSWERS

Problem 1

[/5 marks]

The n^{th} term of an arithmetic sequence is given by $u_n = 15 - 3n$.

(a) State the value of the first term, u_1. $\boxed{u_1 = 12}$ [1]

(b) Given that the n^{th} term of this sequence is -33, find the value of n. [2]

$15 - 3n = -33 \Rightarrow \boxed{n = 16}$

(c) Find the common difference, d. $u_2 = 9$ then $\boxed{d = u_2 - u_2 = 3}$ [2]

Problem 2

[/9 marks]

An *arithmetic* sequence has $u_1 = \log_c(r)$ and $u_2 = \log_c(r^2 s)$

(a) The *constant difference* is $d = u_2 - u_1 = \log_c(r^2 s) - \log_c(r) = \log_c\left(\frac{r^2 s}{r}\right)$

$= \log_c(rs) = \log_c(r) + \log_c(s)$

(b) Let $r = c$ and $s = c^7$ Then $u_1 = \log_c(c) = 1$ and $u_2 = \log_c(c^2 c7) = 9$ and $d = 8$

Therefore $\sum_{n=1}^{5} u_n = \frac{5}{2}(2u_1 + (5-1)d) = \boxed{85}$ [2]

(c) Let $r = c$ and $s = c^7$

$$\sum_{n=6}^{10} u_n = s_{10} - s_5 = \frac{5}{2}(2u_1 + (10-1)d) - \frac{5}{2}(2u_1 + (5-1)d)$$

$$= \frac{5}{2}(2 + 9 \times 8) - \frac{5}{2}(2 + 4 \times 8) = 185 - 85 = \boxed{100}$$

Problem 3

[/7 marks]

Consider the binomial expansion $(x+1)^7 = x^7 + ax^6 + bx^5 + 35x^4 + \ldots + 1$ where $x \neq 0$ and $a, b \in \mathbb{Z}^+$.

(a) Show that $b = 21$. $b = \binom{7}{5} = \frac{7!}{2!5!} = 21$ By the same way: $a = \binom{7}{6} = \frac{7!}{1!6!} = 7$ [2]

The third term in the expansion is the mean of the second term and the fourth term in the expansion.

$7x^2 = \frac{21x^6 + 35x^4}{2} \Rightarrow 14 = 21x^4 + 35x^2 \Rightarrow 3c^2 + 5c - 2 = 0 \quad (c = x^2)$

(b) Find the possible values of x. $c = \frac{-5 \pm 7}{6} \in \{-2; \frac{1}{3}\}$ then $\boxed{x = \sqrt{\frac{1}{3}}}$ [5]

Problem 4

[/6 marks]

Find the least positive value of x for which $\cos\left(\frac{x}{2} + \frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$.

$\frac{x}{2} + \frac{\pi}{3} = \frac{\pi}{4}$ or $\frac{7\pi}{4}$

first positive : $\boxed{x = \frac{17\pi}{6}}$

Problem 5

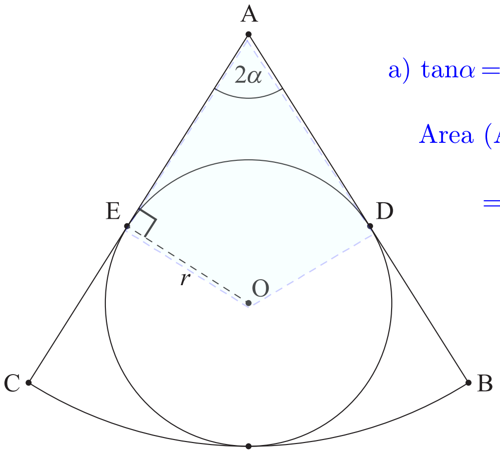
[/9 marks]

The following diagram shows a sector ABC of a circle with centre A. The angle $B\hat{A}C = 2\alpha$, where $0 < \alpha < \frac{\pi}{2}$, and $O\hat{E}A = \frac{\pi}{2}$.

A circle with centre O and radius r is inscribed in sector ABC.

AB and AC are both tangent to the circle at points D and E respectively.

diagram not to scale



a) $\tan\alpha = \frac{r}{AE}$

Area (ADOE) $= 2\frac{AE \times r}{2} = AE \times r$

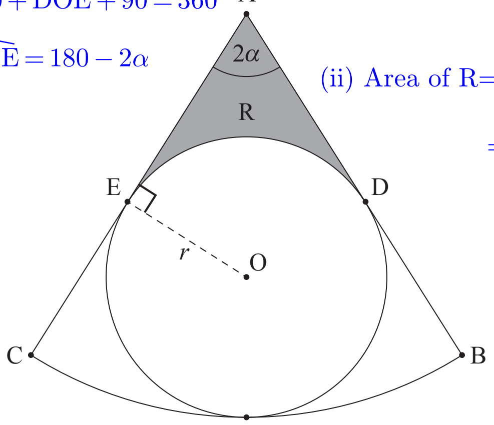
$= \boxed{\frac{r^2}{\tan\alpha}}$

(a) Show that the area of the quadrilateral ADOE is $\dfrac{r^2}{\tan\alpha}$. [4]

(b)

(i) $2\alpha + 90 + \widehat{DOE} + 90 = 360$

$\Rightarrow \widehat{DOE} = 180 - 2\alpha$



(ii) Area of R = Area (ADOE) $- \frac{1}{2}r^2\,\widehat{DOE}$

$= \boxed{\frac{r^2}{\tan\alpha} - \frac{1}{2}r^2(\pi - 2\alpha)}$

(b) (i) Find $D\hat{O}E$ in terms of α.

(ii) Hence or otherwise, find an expression for the area of R. [5]

Problem 6 [/14 marks]

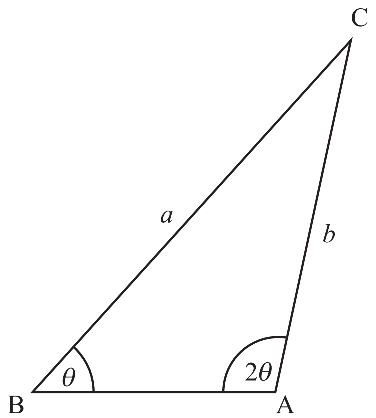
Consider an acute angle θ such that $\cos\theta = \frac{2}{3}$.

(a) Find the value of

(i) $\sin\theta$; $\sin(\theta) = \sqrt{1-\cos^2(\theta)} = \sqrt{1-\frac{4}{9}} = \boxed{\frac{\sqrt{5}}{3}}$

(ii) $\sin 2\theta$. $\sin(2\theta) = 2\sin(\theta)\cos(\theta) = 2\frac{\sqrt{5}}{3}\frac{2}{3} = \boxed{\frac{4\sqrt{5}}{9}}$ [4]

The following diagram shows triangle ABC, with $\hat{\mathrm{B}} = \theta$, $\hat{\mathrm{A}} = 2\theta$, $\mathrm{BC} = a$ and $\mathrm{AC} = b$.



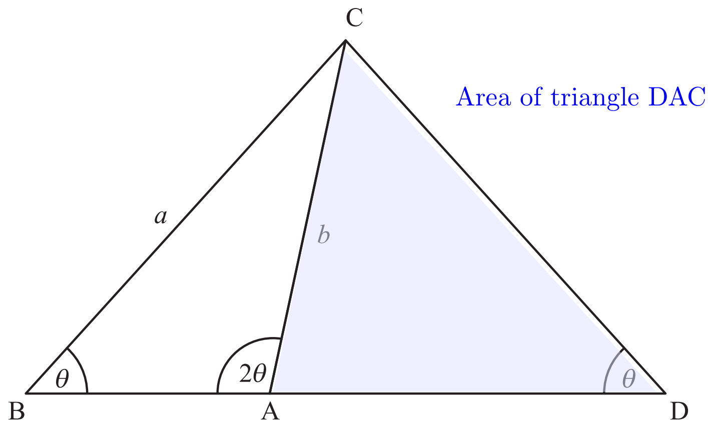
$\frac{b}{\sin(\theta)} = \frac{a}{\sin(2\theta)}$

then $b = \frac{\sin(\theta)a}{\sin(2\theta)}$

$= \frac{\frac{\sqrt{5}}{3}a}{\frac{4\sqrt{5}}{9}} = \frac{3a}{4}$

(b) Show that $b = \frac{3a}{4}$. [2]

$[\mathrm{BA}]$ is extended to form an isosceles triangle DAC, with $\hat{\mathrm{D}} = \theta$, as shown in the following diagram.



Area of triangle DAC $= \frac{1}{2}b^2\sin(\widehat{\mathrm{CAD}})$

(c) Find the value of $\sin\mathrm{C\hat{A}D}$. $\sin(\widehat{\mathrm{CAD}}) = \sin(\pi - 2\theta) = \sin(2\theta) = \frac{4\sqrt{5}}{9}$ [3]

(d) Find the area of triangle DAC, in terms of a. $A = \frac{1}{2}\left(\frac{3a}{4}\right)^2 \cdot \frac{4\sqrt{5}}{9} = \boxed{\frac{\sqrt{5}a^2}{8}}$ [5]

Problem 7 [/13 marks]

Let $f(x) = 2\sin(3x) + 4$ for $x \in \mathbb{R}$.

(a) The range of f is $k \le f(x) \le m$. Find k and m. $\boxed{k=2}$ $\boxed{m=6}$ [3]

Let $g(x) = 5f(2x)$.

(b) Find the range of g. $\boxed{\text{Range}_g = [10, 30]}$ [2]

The function g can be written in the form $g(x) = 10\sin(bx) + c$.

(c) (i) Find the value of b and of c. $\boxed{b=6}$ $\boxed{c=20}$

(ii) Find the period of g. $\boxed{T = \frac{2\pi}{6} = \frac{\pi}{3}}$ [5]

(d) The equation $g(x) = 12$ has two solutions where $\pi \le x \le \frac{4\pi}{3}$. Find both solutions. [3]

2.772 , 2.98704

Problem 8 [/8 marks]

(a) Assuming $\log_9(x) = \log_3(y)$, write y in terms of x.

Help : You could use the identity $\log_a(x) = \frac{\log_b(x)}{\log_b(x)}$ to transform the left member of the equation. [2]

$\log_9(x) = \frac{\log_3(x)}{\log_3(9)} = \frac{\log_3(x)}{2} \Rightarrow \frac{\log_3(x)}{2} = \log_3(y)$ $\log_3(\sqrt{x}) = \log_3(y) \Rightarrow$ $\boxed{y = \sqrt{x}}$

(b) Show that $\log_9(\cos(2x) + 2) = \log_3\left(\sqrt{\cos(2x) + 2}\right)$ [2]

Same way : $\log_9(\cos(2x) + 2) = \frac{\log_3(\cos(2x) + 2)}{2} = \log_3\left(\sqrt{\cos(2x) + 2}\right)$

(c) Hence of otherwise solve $\log_3(2\sin(x)) = \log_9(\cos(2x) + 2)$ for $0 < x < \frac{\pi}{2}$ [4]

By (b) : $\log_3(2\sin(x)) = \log_3\left(\sqrt{\cos(2x) + 2}\right) \Rightarrow 2\sin(x) = \sqrt{\cos(2x) + 2}$

$\Rightarrow 4\sin^2(x) = \cos(2x) + 2$

$\Rightarrow 4\sin^2(x) - \cos(2x) - 2 = 0$

$\Rightarrow 4\sin^2(x) - (2\cos^2(x) - 1) - 2 = 0$

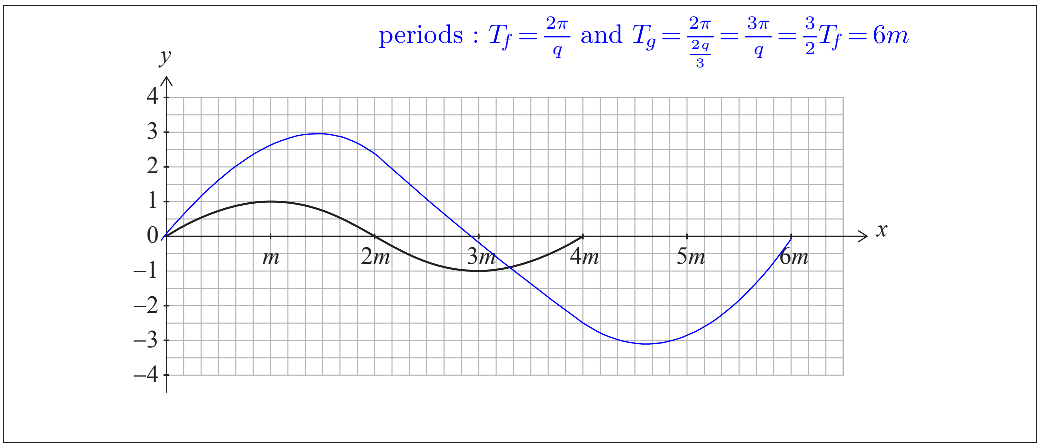
$\Rightarrow 2\cos^2(x) - 1 = 0$

$\Rightarrow \cos^2(x) = \frac{1}{2}$ $\cos(x) = \pm\frac{\sqrt{2}}{2} \Rightarrow$ $\boxed{x = \frac{\pi}{4}}$

Problem 9

[/6 marks]

The function f is defined by $f(x) = \sin qx$, where $q > 0$. The following diagram shows part of the graph of f for $0 \le x \le 4m$, where x is in radians. There are x-intercepts at $x = 0$, $2m$ and $4m$.



(a) Find an expression for m in terms of q. $4m = \frac{2\pi}{q} \Rightarrow \boxed{m = \frac{\pi}{2q}}$ (hence $q = \frac{\pi}{2m}$) [2]

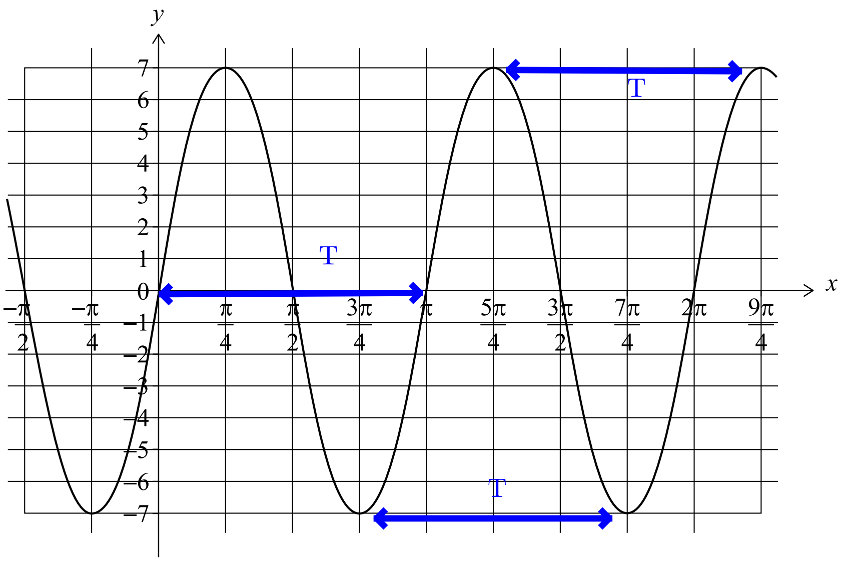
The function g is defined by $g(x) = 3\sin\frac{2qx}{3}$, for $0 \le x \le 6m$.

(b) On the axes above, sketch the graph of g. [4]

Problem 10

[/7 marks]

Consider the function $f(x) = a\sin(bx)$ with $a, b \in \mathbb{Z}^+$. The following diagram shows part of the graph of f.



(a) Write down the value of a. $\boxed{a = 7}$ [1]

(b) (i) Write down the period of f. The period is (here) $\boxed{T = \pi}$ as shown in the picture

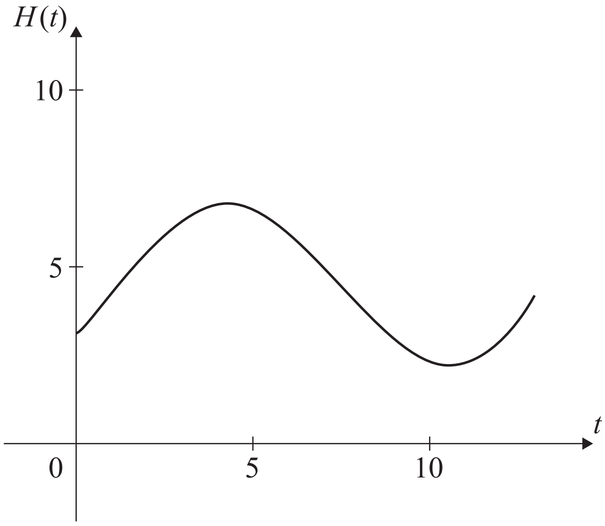
(ii) Hence, find the value of b. $b = \frac{2\pi}{T}$ hence here $\boxed{b = 2}$ [3]

(c) Find the value of $f\left(\frac{\pi}{12}\right)$. $f\left(\frac{\pi}{12}\right) = 7\sin\left(2\frac{\pi}{12}\right) = 7\sin\left(\frac{\pi}{6}\right) = 7\,\frac{1}{2} = \boxed{\frac{7}{2}}$ [3]

Problem 11 [/13 marks]

The height of water, in metres, in Dungeness harbour is modelled by the function $H(t) = a\sin\left(b(t-c)\right) + d$, where t is the number of hours after midnight, and a, b, c and d are constants, where $a > 0$, $b > 0$ and $c > 0$.

The following graph shows the height of the water for 13 hours, starting at midnight.



The first high tide occurs at 04:30 and the next high tide occurs 12 hours later. Throughout the day, the height of the water fluctuates between $2.2\,\mathrm{m}$ and $6.8\,\mathrm{m}$.

All heights are given correct to one decimal place.

(a) Show that $b = \dfrac{\pi}{6}$. That is because $b = \frac{2\pi}{T}$ with $T = 12$ [1]

(b) Find the value of a. $a = \frac{6.8-2.2}{2} = 2.3$ [2]

(c) Find the value of d. $d = \frac{6.8+2.2}{2} = 4.5$ [2]

(d) Find the smallest possible value of c. $H(4.4) = 6.8 \Rightarrow 2.3\sin\left(\frac{\pi}{6}(4.5-c)\right) + 4.5 = \boxed{6.8}$ [3]

$\Rightarrow \boxed{c = 1.5}$

(e) Find the height of the water at 12:00. $\boxed{H(12) = 2.87m}$ [2]

(f) Determine the number of hours, over a 24-hour period, for which the tide is higher than 5 metres. [3]

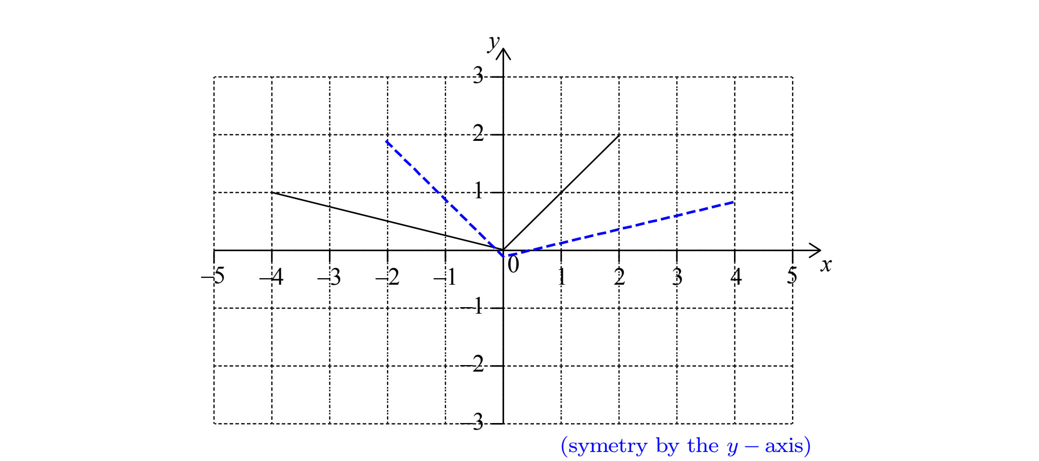
with calculator solve $2.3\sin\left(\frac{\pi}{6}(t-1.5)\right) + 4.5 = 5$

you should get t =1.91852... and t =7.08147 ⇒answer is $\boxed{10.3 \text{ h}}$

Problem 12

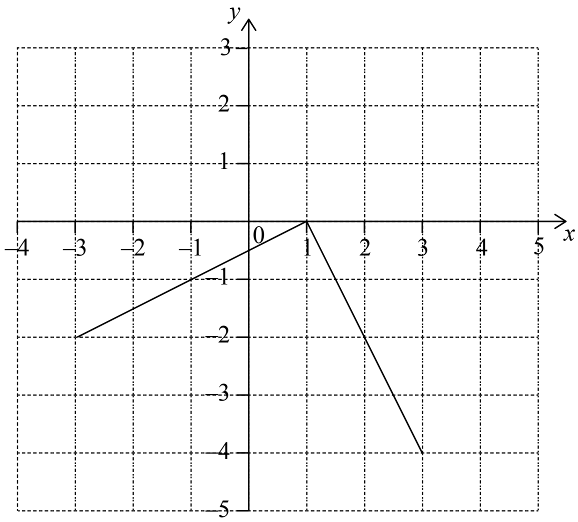
[/6 marks]

The following diagram shows the graph of a function f, for $-4 \leq x \leq 2$.



(a) On the same axes, sketch the graph of $f(-x)$. [2]

(b) Another function, g, can be written in the form $g(x) = a \times f(x+b)$. The following diagram shows the graph of g.



Write down the value of a and of b. $a = -2$ $b = -1$ [4]

Problem 13

[/11 marks]

Let $g(x) = x^2 + bx + 11$. The point $(-1, 8)$ lies on the graph of g. $\Leftrightarrow g(-1) = 8$

(a) Find the value of b. $(-1)^2 + b(-1) + 11 = 8 \Rightarrow -b = 8 - 11 - 1 \quad \Rightarrow \boxed{b = 4}$ [3]

(b) The graph of $f(x) = x^2$ is transformed to obtain the graph of g.

Describe this transformation. $\boxed{\text{translation by the vector } \begin{pmatrix} -2 \\ 7 \end{pmatrix}}$ gives $(x+2)^2 + 7$ [4]

(c) The graph of g is transformed by the two following consecutives transformations to obtain the graph of h : [4]

i) a horizontal stretch of scale factor 2 that gives the curve of $g\left(\frac{x}{2}\right)$

ii) a reflexion by the y-axis. that change the sign of x in the precedent result

Write down the function $h(x)$. $h(x) = g\left(-\frac{x}{2}\right) = \left(-\frac{x}{2}\right)^2 + 4\left(-\frac{x}{2}\right) + 11 = \boxed{\frac{x^2}{4} - 2x + 11}$

Problem 14

[/6 marks]

Let f and g be functions such that $g(x) = 2f(x+1) + 5$.

(a) The graph of f is mapped to the graph of g under the following transformations:

vertical stretch by a factor of k, followed by a translation $\begin{pmatrix} p \\ q \end{pmatrix}$.

Write down the value of

(i) k; $\boxed{2}$

(ii) p; $\boxed{-1}$

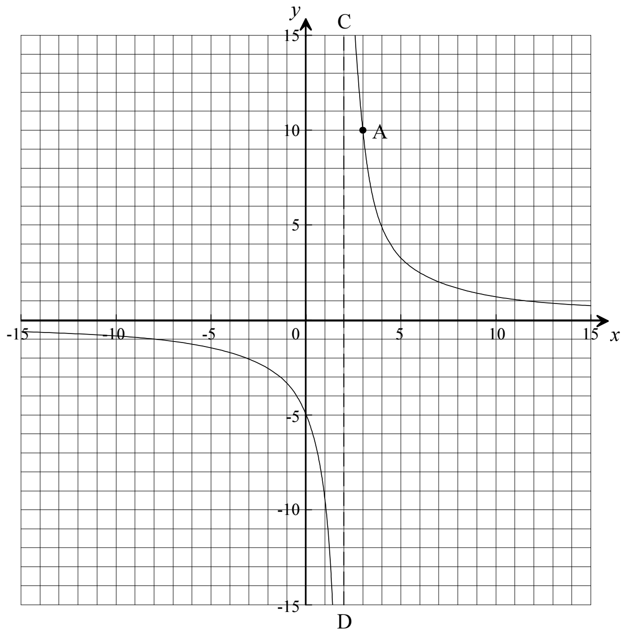
(iii) q. $\boxed{5}$ *[3 marks]*

(b) Let $h(x) = -g(3x)$. The point $A(6, 5)$ on the graph of g is mapped to the point A' on the graph of h. Find A'. $\boxed{A'(2, -5)}$ *[3 marks]*

Problem 15

[/7 marks]

(a) The diagram shows part of the graph of the function $f(x) = \frac{q}{x-p}$. The curve passes through the point $A(3, 10)$. The line (CD) is an asymptote.

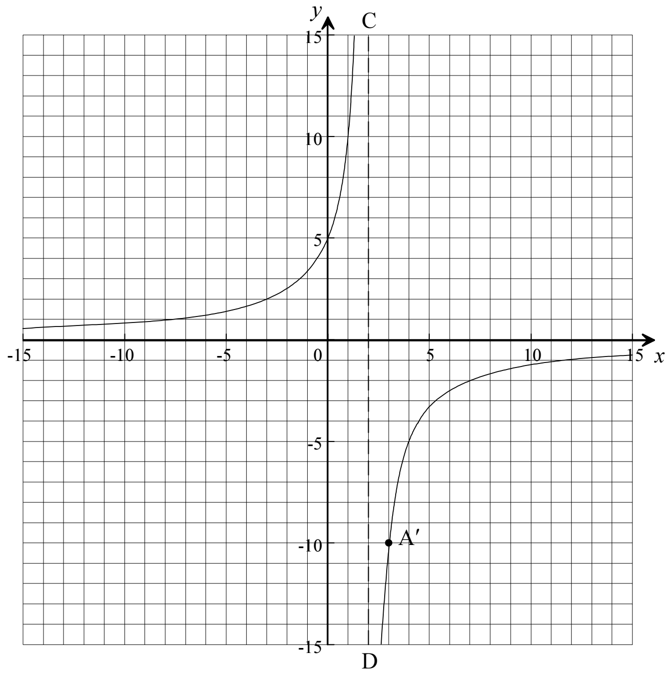


Find the value of

(i) p; $\boxed{p = 2}$ (position of the vertical asymptote)

(ii) q. as $f(3) = 10$ we have: $\frac{q}{3-2} = 10$ then $\boxed{p = 10}$

(b) The graph of $f(x)$ is transformed as shown in the following diagram. The point A is transformed to $A'(3, -10)$.

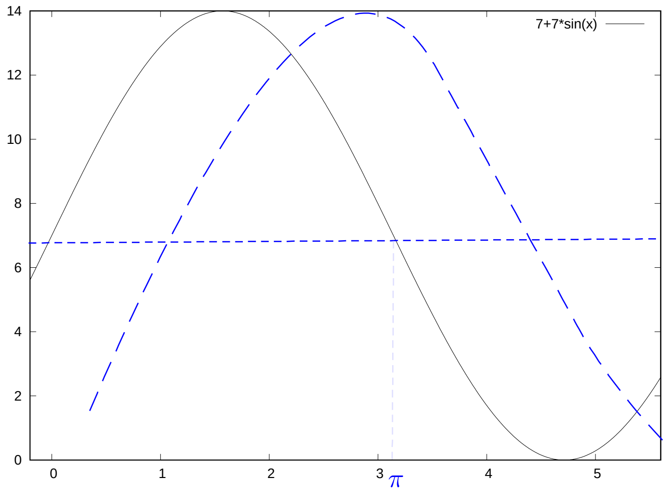


Give a full geometric description of the transformation.

Reflection (axial symetry) by the vertical asymptote $(x = 2)$.

Problem 16 *[/11 marks]*

Let $f(x) = 7 + 7\sin x$. Part of the graph of f is shown below.



(a) The *maximal* value for $f(x)$ is 14 [1]

(b) Solve for $0 \le x < 2\pi$ [5]

(i) $f(x) = 7 \quad \Rightarrow \sin x = 0, \quad x = k\pi \quad \boxed{x = \pi}$

(ii) $f(x) = 0 \quad \Rightarrow \sin x = -1, \quad x = \frac{3\pi}{2} + 2k\pi \quad \boxed{x = \frac{3\pi}{2}}$

(c) Write down the exact value of the x – intercept of f , for $0 \le x < 2\pi$. $\boxed{(\frac{3\pi}{2}, 0)}$ [2]

Let $g(x) = 7 + 7\sin\left(x - \frac{\pi}{2}\right)$.

The graph of f is transformed to the graph of g.

(d) Full geometric description of this transformation : $\boxed{\text{Horizontal translation of } \frac{\pi}{2} \text{ units right}}$