Christmas Examination

Maths AA SL IB$_1$ Part 1

(7 Problems)

Tot: / 42

Tuesday 13 Dec. 2022

* ANSWERS *

You are not permitted access to any calculator for this paper.

Problem 1 / 5marks

The n$^{\text{th}}$ term of an *arithmetic* sequence is given by $u_n = 15 - 3n$.

(a) The value of the first term is $\boxed{u_1 = 12}$ [1]

(b) $u_n = -33 \Rightarrow 15 - 3n = -33 \Rightarrow$ the value of n is $\boxed{n = 16}$ [2]

(c) The common difference, d is $\boxed{-3}$ [2]

Problem 2 / 5marks

(a) The *sum* of these three integers is $3n$, therefore for any integer it is divisible by 3.

(b) The *sum* of the squares of these three integers is $(n-1)^2 + n^2 + (n+1)^3 = 3n^2 + 2$ therefore for any integer it *cannot* be divisible by 3.

Problem 3 / 7marks

Consider the binomial expansion $(x+1)^7 = x^7 + ax^6 + bx^5 + 35x^4 + \ldots + 1$ where $x \neq 0$ and $a, b \in \mathbb{Z}^+$

(a) $(x+1)^7 = \sum_{k=0}^{7} \binom{7}{k} x^k \cdot 1^{7-k} = \sum_{k=0}^{7} \binom{7}{k} x^k$. We get the value of b taking $k = 5$: $b = \binom{7}{5} = \boxed{21}$

(b) The third term in the expansion is the *mean** of the second term and the fourth termin the

In other term : $\binom{7}{5} x^5 = \frac{\binom{7}{6} x^6 + \binom{7}{4} x^4}{2}$

$$\Rightarrow 21x^5 = \frac{7x^6 + 35x^4}{2} \Rightarrow 42x^5 - 7x^6 - 35x^4 = 0 \Rightarrow 6x^5 - x^6 - 5x^4 = 0$$

$$\Rightarrow x^4(6x - x^2 - 5) = 0 \Rightarrow x^4(x-1)(x-5) = 0 \Rightarrow \boxed{x \in \{0, 1, 5\}}$$

Problem 4 / 6marks

(a) $2\cos^2(x) + 5\sin(x) = 4 \Rightarrow 2(1 - \sin^2(x)) + 5\sin(x) - 4 = 0 \Rightarrow 2\sin^2(x) - 5\sin(x) + 2 = 0.$

(b) Hence $2\cos^2(x) + 5\sin(x) = 4 \Rightarrow 2s^2 - 5s + 2 = 0.$ with $s = \sin(x)$

$s = \frac{5 \pm 3}{4} = 2$ or $\frac{1}{2}$ then $\sin(x) = \frac{1}{2}$ $\Rightarrow \boxed{x \in \left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}\right\}}$ for $0 \leqslant x \leqslant 3\pi$.

Problem 5 / 6marks

Find the least positive value of x for which $\cos\left(\frac{x}{2}+\frac{\pi}{3}\right)=\frac{1}{\sqrt{2}}$

Hint : You are supposed to know that $\cos(\alpha)=\frac{1}{\sqrt{2}}$ for $\alpha=\frac{\pi}{4}+2k\pi$ or $\alpha=\frac{7\pi}{4}+2k\pi$

Then $\frac{x}{2}+\frac{\pi}{3}=\frac{\pi}{4}+2k\pi$ or $\frac{x}{2}+\frac{\pi}{3}=\frac{7\pi}{4}+2k\pi$

$\Rightarrow \frac{x}{2}=-\frac{\pi}{12}+2k\pi$ or $\frac{x}{2}=\frac{17\pi}{12}+2k\pi$

$\Rightarrow x=-\frac{\pi}{6}+k\pi=\boxed{\frac{11\pi}{6}+k\pi}$ or $x=\boxed{\frac{17\pi}{6}+k\pi}$ you can also write $\boxed{S=\{\frac{11\pi}{6}+k\pi\}_{k\in\mathbb{Z}}}$

Problem 6 / 6marks

(a) $2x-3-\frac{6}{x-1}=\frac{(2x-3)(x-1)-6}{x-1}=\frac{2x^2-5x-3}{x-1}$ $(x\neq 1)$

(b) Taking $x=\sin(\theta)$ is (a), we get $\frac{2\sin^2(\theta)-5\sin(\theta)-3}{\sin(\theta)-1}=0$ $\Delta=25+24=49$

$x=\frac{5\pm 7}{4}=-\frac{1}{2}$ or $3 \Rightarrow \sin(\theta)=-\frac{1}{2} \Rightarrow \boxed{\theta=\frac{\pi}{6} \text{ or } \theta=\frac{5\pi}{6}}$

Problem 7 / 7marks

Solve $\cos(2x)=5\cos(x)-3$ for $0\leqslant x<2\pi$

$\Rightarrow \cos^2(x)-\sin^2(x)=5\cos(x)-3$

$\Rightarrow \cos^2(x)-(1-\cos^2(x))=5\cos(x)-3$

$\Rightarrow 2\cos^2(x)-5\cos(x)+2=0$

$\Rightarrow 2c^2-5c+2=0$ with $c=\cos(x)$

$\Delta=25-16=9$ $c=\frac{5\pm 3}{4}=2 \text{ or } \frac{1}{2}$ $\Rightarrow \cos(x)=\frac{1}{2} \Rightarrow x=\boxed{\frac{\pi}{3}}$ or $x=2\pi-\frac{\pi}{3}=\boxed{\frac{5\pi}{3}}$ for $0\leqslant x\leqslant 2\pi$.