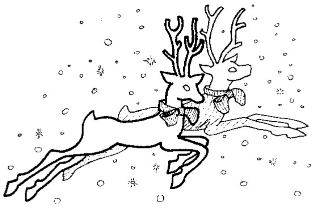

Mathematics HL

IB1 Examination

6 Problems (91 Marks)

Friday 16 December

9:20 to 12:20 pm

*** ANSWERS ***

Problem 1 *[5 marks]*

$3^{(2x-2)} = 3^x - 2 \Leftrightarrow 3^{2x} = 9(3^x - 2) \Leftrightarrow v^2 - 9v + 18 = 0 \Rightarrow v = 3 \text{ or } 6 = 3^x$

$\boxed{x \in \{\ 1\ ,\ 1 + \log_3(2)\ \}}$

Problem 2 *[8 marks]*

i) P(2): 2=2 P(3) : 8=8 P(4): 20=20

ii) P(5): $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 = \frac{1}{3}(4 \cdot 5 \cdot 6)$ true (40 = 40)

iii) Eq(n): $\displaystyle\sum_{k=1}^{n} k(k+1) = \frac{1}{3}(n(n+1)(n+2)$

iv) We already know Eq(n) true for $n = 1$ (initial value).

To see: Eq(m) $\Rightarrow$ Eq($m+1$) for any integer $m \geqslant 1$

Then assuming Eq(m): $\displaystyle\sum_{k=1}^{m} k(k+1) = \frac{1}{3}(m(m+1)(m+2)$ for a certian $m \in \mathbb{N}$.

We want to show Eq($m+1$): $\displaystyle\sum_{k=1}^{m+1} (k+1)(k+2) = \frac{1}{3}(m+1)(m+2)(m+3)$

that is: $\displaystyle\sum_{k=1}^{m} k(k+1) + (m+1)(m+2) \stackrel{?}{=} \frac{1}{3}(m+1)(m+2)(m+3)$

$\frac{1}{3}(m(m+1)(m+2) + (m+1)(m+2) = (m+1)(m+2)(\frac{m}{3} + 1) = \frac{1}{3}(m+1)(m+2)(m+3)$

$\Rightarrow$ ok

Problem 3 *[6 marks]*

1) Using Pascal's Triangle (or otherwise) , give the expansiont of $(a+b)^5$ and of $(a-b)^5$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

2) $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

$(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$

$\sum$: $2a^5 + 20a^3b^2 + 10ab^4$

3) $(a+b)^5 + (a-b)^5 = \boxed{(2a)(a^4 + 10a^2b^2 + 5b^4)}$

Complementary question: can you find a better factorisation ?

Problem 4 *[7 marks]*

Le us consider the sequence $\{u_n\}$ having first terms $-2, -1, 3, 13, 35$ .... (starting with $n = 1$)

i) This sequence is neither *arithmetic*, nor *geometric*.

ii) & iii)

n	u_n (as given)	$a_n = 3 + 2n$	$g_n = u_n + a_n$
1	-2	5	3
2	-1	7	6
3	3	9	12
4	13	11	24
5	35	13	48
$\vdots$			
n			$3 \cdot 2^{n-1}$

iv) $s_n = g_1 \frac{r^n - 1}{r - 1}$

$$\sum_{k=5}^{10} g_k = s_{10} - s_4 = g_1\left(\frac{r^{10}-1}{r-1} - g_1\frac{r^4-1}{r-1}\right) = 3\left(\frac{2^{10}-2^4}{2-1}\right) = 3 \times 2^4(2^6 - 1) = 3 \times 16 \times 63 = \boxed{3024}$$

Problem 5 *[6 marks]*

Let us consider the complex numbers $z_1 = 4e^{i\frac{\pi}{6}}$ and $z_2 = \sqrt{2}e^{i\frac{3\pi}{4}}$

1) $z_1 = 4\text{cis}\left(\frac{\pi}{6}\right) = 4\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \boxed{2\sqrt{3} + 2i}$ and $z_2 = \sqrt{2}\text{cis}\left(\frac{3\pi}{2}\right) = \sqrt{2}\left(\frac{-\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \boxed{-1 + i}$

2) The product $z_1^3 z_2^8$ is $(64i)(16) = \boxed{1024i}$

3) $\left(2\frac{z_2}{z_1}\right)^2 = \boxed{-16\sqrt{3} + 16i}$

4) Show that $\sqrt{i} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$

5) $\sqrt{z_1^3 z_2^8} = \sqrt{1024i} = \sqrt{1024}\sqrt{i} = 32\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = \boxed{16\sqrt{2} + 16\sqrt{2}i}$

Problem 6 *[7 marks]*

1) Let us consider $z = \frac{i}{2}$.

- $|z| = \frac{1}{2}$
- The *argument* of z is $\boxed{\theta = \frac{\pi}{2}}$
- Rhe *conjugate* of z is $\boxed{z^* = -\frac{i}{2}}$
- $z^2 = \boxed{-\frac{1}{4}}$, $z^3 = \boxed{-\frac{i}{8}}$ and $z^4 = \boxed{\frac{1}{16}}$
- $\frac{1}{1-z} = \frac{1}{1-\frac{i}{2}} = \frac{2}{2-i} = \frac{2(2+i)}{(2-i)(2+i)} = \frac{4+i}{2+1} = \boxed{\frac{4}{5} + \frac{2}{5}i}$

2) Show that, for $|x| < 1$

$1 + x + x^2 + x^3 + x^4 + \ldots$ converges to $\frac{1}{1-x}$

$1 + x + x^2 + x^3 + x^4 + \ldots x^n = \frac{1 - x^{n+1}}{1-x}$ and $\lim_{n \to \infty} x^{n+1} = 0$ (if $|x| < 1$)

$$\text{then } \lim_{n \to \infty} \frac{1 - x^{n+1}}{1-x} = \frac{1}{1-x}$$

3) We will admit that $1 + z + z^2 + z^3 + z^4 +$ converge the same way for any $z \in \mathbb{C}$, if $|z| < 1$.

An expansion for $\frac{1}{1-z}$ with 4 terms is $\frac{1}{1-z} = 1 + z + z^2 + z^3$

$$= 1 + \left(i\frac{1}{2}\right) + \left(i\frac{1}{2}\right)^2 + \left(i\frac{1}{2}\right)^3$$

$$= \boxed{1 + \frac{i}{2} + \frac{-1}{4} + \frac{-i}{8}}$$

. Simplify your precedant answer, to give it the form $a + ib$

$$= 1 - \frac{1}{4} + i\left(\frac{1}{2} - \frac{1}{8}\right) = \boxed{\frac{3}{4} + \frac{3}{8}i}$$

. Consider the *limit* when the number of terms in the expansion becomes *infinite* and write the result on the form $a + ib$

$$\frac{1}{1-z} = 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{32} + \cdots + i\left(\frac{1}{2} - \frac{1}{8} + \frac{1}{32} - \frac{1}{64} + \cdots + \right)$$

$$= \left(1 \frac{1}{1 - \left(\frac{1}{4}\right)}\right) + i\left(\frac{1}{2} \frac{1}{1 - \left(\frac{1}{4}\right)}\right) = \frac{1}{\frac{5}{4}} + i\left(\frac{1}{2}\frac{1}{\frac{5}{4}}\right) = \boxed{\frac{4}{5} + \frac{2}{5}i}$$ as already found in (1)