Christmas Examination

Maths AA SL IB$_1$ Part 2

(7 Problems)

Tot: / 52

Tuesday 10 Dec.2024

ANSWERS

Question 1 [15 marks]

Let us consider the sequence u: $\frac{15}{4};\frac{3}{2};\frac{3}{5}\dots$

1) Is is *arithmetic* or *geometric* ? [1]

2) What is the first term u_n smaller than $\frac{1}{160}$? [4]

3) Let us consider the series $S_n = \sum_{k=1}^{n} u_k$ [4]

S_n can be written on the form $\frac{a}{b}\left(1-\left(\frac{c}{d}\right)^n\right)$. Find a, b, c, d

4) Give the value of $\sum_{k=4}^{8} u_k$ [4]

5) What gives S_n if n is *infinite* ? [2]

Answers

1) *Geometric*: $\frac{3}{2}\frac{4}{15} = \frac{3}{5}\frac{2}{3} = \boxed{\frac{2}{5}}$

2) We solve $\frac{15}{4}\left(\frac{2}{5}\right)^{n-1} = \frac{1}{160} \Rightarrow n-1 = \log_{\left(\frac{2}{5}\right)}\left(\frac{1}{160}\frac{4}{15}\right) = \log_{\left(\frac{2}{5}\right)}\left(\frac{1}{600}\right) = 6.98 \quad \Rightarrow \boxed{n=8}$

the first term is $\boxed{u_7 = \frac{15}{4}\left(\frac{2}{5}\right)^7 \cong 0.006144}$ $\qquad u_7 < \frac{1}{160} \cong 0.00625$

3) $S_n = \frac{15}{4}\cdot\frac{\left(\frac{2}{5}\right)^n - 1}{\left(\frac{2}{5}\right)-1} = \frac{15}{4}\cdot\frac{1-\left(\frac{2}{5}\right)^n}{1-\left(\frac{2}{5}\right)} = \frac{15}{4}\cdot\frac{1-\left(\frac{2}{5}\right)^n}{\frac{5-2}{5}} = \frac{15}{4}\left(\frac{5}{3}\right)\left(1-\left(\frac{2}{5}\right)^n\right) = \boxed{\frac{25}{4}\left(1-\left(\frac{2}{5}\right)^n\right)}$

then: $\frac{a}{b}\left(1-\left(\frac{c}{d}\right)^n\right) = \frac{25}{4}\left(1-\left(\frac{2}{5}\right)^n\right)$ for $\boxed{a=25, b=4, c=2, d=5}$

4) $\sum_{k=4}^{8} u_k = S_8 - S_3 = \frac{25}{4}\left(1-\left(\frac{2}{5}\right)^8\right) - \frac{25}{4}\left(1-\left(\frac{2}{5}\right)^3\right) = \frac{25}{4}\left(\left(\frac{2}{5}\right)^3 - \left(\frac{2}{5}\right)^8\right) = \boxed{\frac{2}{5}\left(1-\left(\frac{2}{5}\right)^5\right)}$

5) $S_\infty = \frac{25}{4}\left(1-\left(\frac{2}{5}\right)^\infty\right) = \boxed{\frac{25}{4}}$

Question 2 [15 marks]

Two friends Amelia and Bill, each set themselves a target of saving $\$20\,000$. They each have $\$9000$ to invest.

(a) Amelia invests her $\$9000$ in an account that offers an interest rate of 7% per annum compounded **annually**.

(i) Find the value of Amelia's investment after 5 years to the nearest hundred dollars.

(ii) Determine the number of years required for Amelia's investment to reach the target. [5]

(b) Bill invests his $\$9000$ in an account that offers an interest rate of $r\%$ per annum compounded **monthly**, where r is set to two decimal places.

Find the minimum value of r needed for Bill to reach the target after 10 years. [3]

(c) A third friend Chris also wants to reach the $\$20\,000$ target. He puts his money in a safe where he does not earn any interest. His system is to add more money to this safe each year. Each year he will add half the amount added in the previous year.

(i) Show that Chris will never reach the target if his initial deposit is $\$9000$.

(ii) Find the amount Chris needs to deposit initially in order to reach the target after 5 years. Give your answer to the nearest dollar. [8]

(a)

i) $9000 \times \left(1+\frac{7}{100k}\right)^{5k}$ (with $k=1$)

$= \boxed{12623}$

ii) $2000 = 9000 \times \left(1+\frac{r}{100k}\right)^{nk}$ (with $k=1$ and $r=7$)

$\Rightarrow n = \dfrac{\log\left(\frac{20}{9}\right)}{\log\left(\frac{107}{100}\right)} \simeq \boxed{11.8 \text{ years}}$

(b) IB's Marksheme

METHOD 1

attempt to substitute into compound interest formula (condone absence of compounding periods) ***(M1)***

$$9000\left(1+\frac{r}{100\times 12}\right)^{12\times 10} = 20000$$

$8.01170...$ ***(A1)***

$r = 8.02\ (\%)$ ***A1***

METHOD 2

$n = 10$
$\text{PV} = \pm 9000$
$\text{FV} = \mp 20000$
$\text{P/Y} = 1$
$\text{C/Y} = 12$
$r = 8.01170...$ ***(M1)(A1)***

Note: Award ***M1*** for an attempt to use a financial app in their technology, award ***A1*** for $(r=)\ 8.01170\ldots$

$r = 8.02\ (\%)$ ***A1***

(c)

(i) recognising geometric series (seen anywhere) ***(M1)***

$$r = \frac{4500}{9000}\ \left(=\frac{1}{2}\right)$$ ***(A1)***

EITHER

considering S_∞ ***(M1)***

$$\frac{9000}{1-0.5}\ (=18000)$$ ***A1***

correct reasoning that $18000 < 20000$ ***R1***

Question 3 [8 marks]

Consider the expansion of $\left(3x^2 - \frac{k}{x}\right)^9$, where $k > 0$.

The coefficient of the term in x^6 is 6048. Find the value of k.

Consider the expansion of $\left(2x^3 - \frac{k}{x^2}\right)^9$ where $k > 0$

The coefficient of the *term* in x^{17} is 41472

Find k

Answers

$$\left(2x^3 - \frac{k}{x^2}\right)^9 = \sum_{n=0}^{9} \binom{9}{k} (2x^3)^n \left(-\frac{k}{x^2}\right)^{9-n}$$

For having x^{17}: $\frac{x^{3n}}{x^{2(9-n)}} = x^{17}$ $\Rightarrow 3n - 18 + 2n = 17$ $\Rightarrow 5n = 35$ $\Rightarrow n = 7$

Therefore the *term* in x^{17} is :

$$\binom{9}{7} (2x^3)^7 \left(-\frac{k}{x^2}\right)^{9-7} = 36 \times 2^7 x^{21} k^2 x^{-4} = 4608 k^2 x^{17}$$

$\Rightarrow 4608k^2 = 41472$ $\quad k = \sqrt{\frac{41472}{4608}} = \boxed{3}$

Question 4 [8 marks]

The expansion of $(x+h)^8$, where $h > 0$, can be written as $x^8 + ax^7 + bx^6 + cx^5 + dx^4 + ... + h^8$, where $a, b, c, d, ... \in \mathbb{R}$.

(a) Find an expression, in terms of h, for

(i) a;

(ii) b;

(iii) d. [4]

(b) Given that a, b, and d are the first three terms of a geometric sequence, find the value of h. [3]

Answers

(a) $(x+h)^8 = \sum_{k=0}^{8} \binom{8}{k} h^{8-k} x^k$

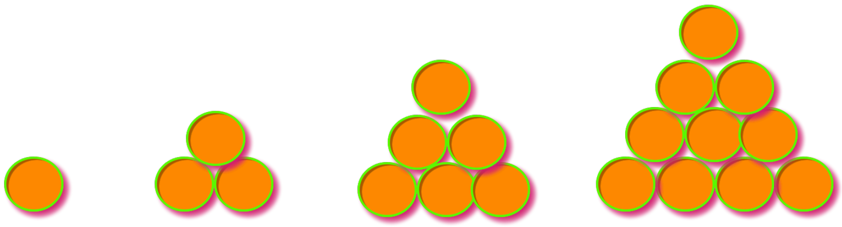
then $a = \binom{8}{8-7} h^{8-7} = \boxed{8h}$ $\quad b = \binom{8}{8-6} h^2 = 28h^2$ $\quad d = \binom{8}{4} h^4 = 70h^4$

(b) $u_1 = a$ $\quad u_2 = u_1 \times r = 28h^2$ $\quad u_3 = u_2 \times r = 70h^4$

then $r = \frac{u_2}{u_1} = \frac{u_4}{u_2} \Rightarrow \frac{b}{a} = \frac{d}{b} \Rightarrow \frac{28h^2}{8h} = \frac{70h^4}{28h^2} \Rightarrow \frac{7}{2}h = \frac{5}{2}h^2 \Rightarrow \boxed{h = \frac{7}{5}}$

Question 5 [8 marks]

A child received a box containing 100 small discs for his birthday. He arranges them into groups of increasing size. The figure below shows the first 4 groups.



1) The number of boxes the group number n is $\frac{n(n+1)}{2}$

The following table sows in the second column the number of disk of group n

and in the third culumn the total number of disk required.

n	n_{group}	$\text{tot}(n)$
1	1	1
2	3	4
3	6	10
4	10	20
5	15	35
6	21	56
7	28	84
8	36	120
9	45	165
10	55	220
11	66	286
12	78	364
13	91	455
14	105	560

As we can see, we need more than 100 disk for completing 8 groups, and can complete only 7 groups (using 84 disks).

2) The to obtain the double (i.e 17 groups) we need 560 disks, therefore 6 boxes
$\Rightarrow$ $\boxed{\text{5 additional boxes}}$

Notice : HL students can show that

$$\text{tot}(n) = \sum_{k=1}^{n} \frac{k(k+1)}{2} = \frac{n(n+1)(n+2)}{6}$$