Christmas Examination

Maths AA SL IB_1 Part 2

(8 Problems)

Tot: / 60

Tuesday 12 December 2023

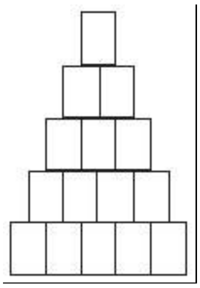
* ANSWERS *

A calculator is allowed for this paper

Problem 1

[/14 marks]

Natasha organizes cans in triangular piles, where each row has one less can than the row below. For example, the pile of 15 cans shown has 5 cans in the bottom row and 4 cans in the row above it.



To simplify we will always start by the top.
for example 1+2+3+4+5=15 cans

$u_1 = 1 \quad d = 0$

$S_n = \sum_{k=1}^{n} u_k \; \frac{n}{2}(2+(n-1)) = \frac{n(n+1)}{2}$

(a) A pile has 20 cans in the bottom row. Show that the pile contains 210 cans. **(4)**

$\frac{20 \times 21}{2} = 210$

(b) There are 3240 cans in a pile. How many cans are in the bottom row? **(4)**

$\frac{n(n+1)}{2} = 3240 \quad \Rightarrow \; n^2 + n - 6480 = 0 \Rightarrow \boxed{n = 80}$

(c) (i) There are S cans and they are organized in a triangular pile with n cans in the bottom row. Show that $n^2 + n - 2S = 0$. $\frac{n(n+1)}{2} = S \Rightarrow \; n^2 + n - 2S = 0$

(ii) Natasha has 2100 cans. Explain why she cannot organize them in a triangular pile. **(6)**

$n^2 + n - 2(2100) = 0 \qquad \Delta = 9601 \qquad \sqrt{\Delta} \notin \mathbb{N}$

Problem 2

[/6 marks]

The following table shows four series of numbers. One of these series is geometric, one of the series is arithmetic and the other two are neither geometric nor arithmetic.

(a) Complete the table by stating the type of series that is shown.

Series		Type of series
(i)	$1+11+111+1111+11111+\ldots$	—
(ii)	$1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+\ldots$	*geometric* $r=\frac{3}{4}$
(iii)	$0.9+0.875+0.85+0.825+0.8+\ldots$	*arithm.* $d=-0.025$
(iv)	$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\ldots$	—

(b) The geometric series can be summed to infinity. Find this sum. $S_\infty = 1\frac{1}{1-\frac{3}{4}} = \boxed{4}$

Problem 3

[/5 marks]

An arithmetic sequence has first term 60 and common difference -2.5.

(a) Given that the kth term of the sequence is zero, find the value of k. [2]

(a) $60+(k-1)(-\frac{5}{2})=0$

$\Rightarrow \boxed{k=25}$

Let S_n denote the sum of the first n terms of the sequence.

(b) Find the maximum value of S_n. $S_{max}=S_{25}=\frac{25}{2}(120+24\times(-\frac{5}{2}))=\boxed{2250}$ [3]

Problem 4

[/9 marks]

The sum of the first n terms of a geometric sequence is given by $S_n=\sum_{r=1}^{n}\frac{2}{3}\left(\frac{7}{8}\right)^r$.

(a) Find the first term of the sequence, $u_1=s_1=\frac{2}{3}\cdot\frac{7}{8}=\boxed{\frac{7}{12}}$ [2]

(b) Find S_∞. $=\frac{2}{3}\frac{1}{1-\frac{7}{8}}=\frac{2}{3}8=\boxed{\frac{16}{3}}$ [3]

(c) Find the least value of n such that $S_\infty - S_n < 0.001$. [4]

Problem 5

[/6 marks]

Consider the expansion of $(3+x^2)^{n+1}$, where $n\in\mathbb{Z}^+$.

$$\sum_{k=0}^{n+1} {}_{n+1}\mathcal{C}_k 3^{n+1-k}(x^2)^k$$

Given that the coefficient of x^4 is 20412, find the value of n.

${}_{n+1}\mathcal{C}_2\, 3^{n-1}=20412 \quad \Rightarrow \frac{n(n+1)}{2}3^{n-1}=20412 \qquad \boxed{n=7}$ for having x^4 we take $k=2$

Problem 6

[/7 marks]

In an arithmetic sequence,

the term $u_{40}=106$ and the sum $S_{40}=1900$.

(a) $u_1+(40-1)d=106$ then $(40-1)d=106-u_1$

$\frac{40}{2}(2u_1+(40-1)d)=1900$ then $20(2u_1+106-u_1)=1900$

$\Rightarrow u_1+106=95 \quad \Rightarrow \boxed{u_1=-11 \quad \text{and} \quad d=3}$

(b) The *general term* u_n is $u_n=-11+(n-1)3 \quad =\boxed{3n-14}$

Problem 7

[/6 marks]

Consider the expansion of $\left(3x^2-\frac{k}{x}\right)^9$, where $k>0$.

The coefficient of the term in x^6 is 6048. Find the value of k.

$$\sum_{j=0}^{9} {}_9\mathcal{C}_j\,(3x^2)^j(-k)^{9-j}(x^{-1})^{9-j}=\sum_{j=0}^{n} {}_9\mathcal{C}_j 3^j(-k)^{9-j}x^{3j-9}$$ taking $k=5$: ${}_9\mathcal{C}_5 3^5(-k)^4=6048$

$\boxed{k=\frac{2}{3}}$

Problem 8 [/7 marks]

The coefficient of x^6 in the expansion of $(ax^3+b)^8$ is 448.

The coefficient of x^6 in the expansion of $(ax^3+b)^{10}$ is 2880.

Find the value of a and the value of b, where $a, b > 0$.

$_8\mathcal{C}_k(ax^3)^k\, b^{8-k}$ taking $k=2$ $\quad {_8\mathcal{C}_2}(ax^3)^2\, b^6 = \frac{8!}{6!2!}a^2b^6\, x^6 \Rightarrow \frac{7\times 8}{2}a^2b^6 = 448$

$_{10}\mathcal{C}_k(ax^3)^k\, b^{10-k}$ taking $k=2$ $\quad {_{10}\mathcal{C}_2}(ax^3)^2\, b^8 = \frac{10!}{8!2!}a^2b^8\, x^6 \Rightarrow \frac{9\times 10}{2}a^2b^8 = 2880$

Then a and b are solutions of

$$\begin{cases} a^2b^6 = \frac{448}{28} = 16 \\ a^2b^8 = \frac{2880}{45} = 64 \end{cases} \Rightarrow \frac{a^2b^8}{a^2b^6} = \frac{64}{16} \quad \Rightarrow b^2 = 4 \quad \Rightarrow \boxed{b=2}$$