Christmas Examination

Maths AA SL IB$_1$ Part 1

(7 Problems)

Tot: / 40

Tuesday 12 December 2023

Name : _____________

You are <u>not</u> permitted access to any calculator for this paper.

Problem 1

[/6 marks]

Solve the following equations:

i) $\dfrac{x(17-2x)-1}{5}=4 \quad \Rightarrow 2x^2-17x+21=0 \Rightarrow 2(x-\frac{3}{2})(x-7)=0 \qquad S=\{\frac{3}{2},7\}$

ii) $\dfrac{\log_3(x)(17-\log_3(x^2))-1}{5}=\log_2(16)$

$\Rightarrow 17\log_3(x)-2\log_3^2(x)-1=5\times 4 \Rightarrow 2l^2-17l+21=0 \quad \text{with } l=\log_3(x)$

then by (i) : $\log_3(x)\in\{\frac{3}{2},7\} \quad \Rightarrow \boxed{x=3^{\frac{3}{2}}=\sqrt{27} \quad \text{or} \quad x=3^7}$

Problem 2

[/6 marks]

(a) Calculate the value of each of the following logarithms:

(i) $\log_2\dfrac{1}{16}$; $=-4$

(ii) $\log_9 3$; $=\frac{1}{2}$

(iii) $\log_{\sqrt{3}} 81$. $=4$

(b) It is given that $\log_{ab} a=3$, where $a, b\in\mathbb{R}^+$, $ab\neq 1$.

(i) Show that $\log_{ab} b=-2$.

(ii) Hence find the value of $\log_{ab}\dfrac{\sqrt[3]{a}}{\sqrt{b}}$.

(b) (i) $\log_{ab}(a)=3 \Leftrightarrow (ab)^3=a$

$\Leftrightarrow a^3b^3=a$

$\Leftrightarrow a^2b^3=1$

$\Leftrightarrow b^3=\frac{1}{a^2}=a^{-2}$

$\Leftrightarrow b=\sqrt[3]{a^{-2}}=a^{-\frac{2}{3}}$

therefore $\log_{ab}b=\log_{ab}\left(a^{-\frac{2}{3}}\right)$

$=-\frac{2}{3}\log_{ab}(a)$

$=-\frac{2}{3}\,3=-2$

(ii) Hence $\log_{ab}\left(\frac{3\sqrt{a}}{\sqrt{b}}\right)$

$=\log_{ab}(3\sqrt{a})-\log_{ab}(\sqrt{b})$

$=\frac{1}{3}\log_{ab}(a)-\frac{1}{2}\log_{ab}(b)$

$=\frac{1}{3}\times 3-\frac{1}{2}\times(-2)=\boxed{2}$

Problem 3

[/4 marks]

Let us consider $a=\log_3(2)$ and $b=\log_3(5)$, $x=\log_3(1+\frac{3}{125})$ and $y=\log_3(100)$.

Then $x=\log_3(\frac{128}{125})=\log_3(128)-\log_3(128)=\log_3(2^7)-\log_3(5^3)=\boxed{7a-3b}$

$y=\log_3(4\times 25)=\log_3(2^2)+\log_3(5^2)=\boxed{2a+2b}$

Problem 4 [/5 marks]

The n$^{\text{th}}$ term of an *arithmetic* sequence is given by $u_n = 15 - 3n$.

(a) The value of the first term is : $u_1 = 15 - 3 = \boxed{12}$

(b) $u_n = 15 - 3n = -33 \Leftrightarrow 3n = 48 \Rightarrow \boxed{n = 16}$

(c) The *common difference* is $\boxed{d = -3}$

Problem 5 [/6 marks]

An *geometric* sequence has first term $u_1 = a$ and second term $u_2 = a^2 - 3a$, where $a > 0$.

(a) We fin *constant ratio* (in terms of a) by considering $r = \frac{u_2}{u_1} = \boxed{a - 3}$.

Let us consider the series $s_n = \sum_{k=1}^{n} u_k$.

(b) The *general term* for s_n is : $s_n = u_1 \frac{r^n - 1}{r - 1} = a\frac{(a-3)^n - 1}{(a-3) - 1} = a\frac{(a-3)^n - 1}{a - 4}$

(c) We vant all the values of a for which the sum to infinity exists.

Hint : There is a formula about this condition in the IB booklet, that is : $|r| < 1$

then $|a - 3| < 1 \Leftrightarrow -1 < (a - 3) < 1 \Leftrightarrow -1 < (a - 3) < 1 \Leftrightarrow 2 < a < 4 \Leftrightarrow \boxed{a \in (2, 4)}$

Problem 6 [/7 marks]

The expansion of $(x + h)^8$, where $h > 0$, can be written as $x^8 + ax^7 + bx^6 + cx^5 + dx^4 + ... + h^8$, where $a, b, c, d, ... \in \mathbb{R}$.

$x^8 + 8x^7h + 28x^6h^2 + 56x^5h^3 + 70x^4h^4 + 56x^3h^5 + 28x^2h^6 + 8xh^7 + h^8$

(a) Find an expression, in terms of h, for

(i) a; $= 8$

(ii) b; $= 28h^2$

(iii) d. $= 70h^4$ [4]

(b) $r = \frac{c}{b} = \frac{b}{a}$ then $\frac{56h^3}{28h^2} = \frac{28h^2}{8} \Rightarrow 2h = \frac{7h^2}{2} \Rightarrow \boxed{h = \frac{4}{7}}$

(b) Given that a, b, and d are the first three terms of a geometric sequence, find the value of h. [3]

Problem 7 [/6 marks]

Consider the binomial expansion $(x+1)^7 = x^7 + ax^6 + bx^5 + 35x^4 + ... + 1$

where $x \neq 0$, $a \neq 0$, $b \neq 0$, $a, b \in \mathbb{N}$ $(x+1)^7 = x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$

(a) Show that $b = 21$ $b = {}_7\mathcal{C}_5$

(b) The third term in the expansion is the *mean* of the second term and the fourth term in the expansion. Find the possible values of x. $21x^5 = \frac{7x^6 + 35x^4}{2} \Rightarrow x^2 - 6x + 5 = 0$ $\boxed{S = \{1, 5\}}$