Christmas Examination

Maths SL IB$_2$

Part 1

(8 Problems 83 marks)

Monday 15 Dec. 2025
Duration : 90 min

ANSWERS

Problem 1 [/ 4 marks]

Consider the function $f(x) = x^2 + 5x - 8$, where $x \in \mathbb{R}$

(a) $f'(x) = 2x + 5 \Rightarrow \boxed{f'(1) = 7}$ [2]

(b) The equation of the tangent to the graph of f at $x = 1$. [2]

is $y = ax + b$ with $a = f'(1) = 7$, then $y = 7x + b$

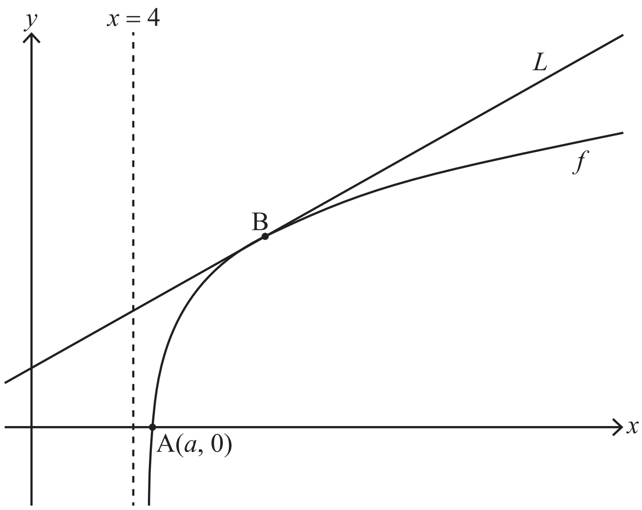
As it passes through $(1, -2)$ then $-2 = 7 + b \Rightarrow b = -9 \Rightarrow \boxed{y = 7x - 9}$

Problem 2 *please see next page*

Problem 3 [/ 9 marks]

Consider the function f defined by $f(x) = \ln(x^2 - 16)$ for $x > 4$.

The following diagram shows part of the graph of f which crosses the x-axis at point A, with coordinates $(a, 0)$.The line L is the tangent to the graph of f at the point B.



(a) Find the exact value of a. $y = 0$ for $x^2 - 16 = 1 \Rightarrow x = \pm\sqrt{17}$, then $\boxed{a = \sqrt{17}}$ [3]

(b) Given that the gradient of L is $\frac{1}{3}$, find the x-coordinate of B. [6]

$f'(x) = \frac{2x}{x^2 - 16} = \frac{1}{3} \Rightarrow x^2 - 6x - 16 = 0 \quad \Delta = 36 + 64 = 100 \quad x_B = 8 \qquad \boxed{B: (8, \ln(48))}$

Problem 2 [/ 4 marks]

The derivative of a function g is given by $g'(x) = \cos x + e^{2x}$, where $x \in \mathbb{R}$.

Given that $g(0) = 7$, find $g(x)$.

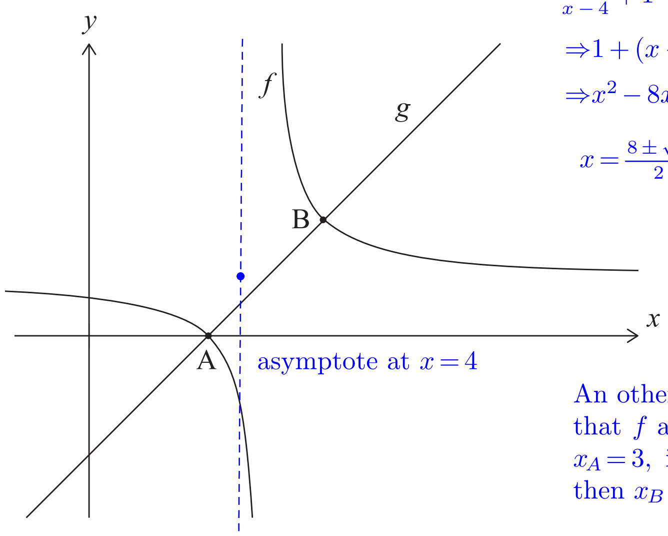
$$g(x) = \int (\cos(x) + e^{2x})dx = \sin(x) + \frac{e^{2x}}{2} + c$$

As it passes through $(0,7)$ then $7 = \sin(0) + \frac{e^0}{2} = \frac{1}{2} \Rightarrow c = \frac{13}{2} \Rightarrow \boxed{g(x) = \sin(x) + \frac{e^{2x}}{2} + \frac{13}{2}}$

Problem 4 [/ 15 marks]

Consider the functions $f(x) = \dfrac{1}{x-4} + 1$, for $x \neq 4$, and $g(x) = x - 3$ for $x \in \mathbb{R}$.

The following diagram shows the graphs of f and g.



$\frac{1}{x-4} + 1 = x - 3$

$\Rightarrow 1 + (x-4) = (x-3)(x-4)$

$\Rightarrow x^2 - 8x + 15 = 0$

$x = \frac{8 \pm \sqrt{4}}{2} = 3 \text{ or } 35$

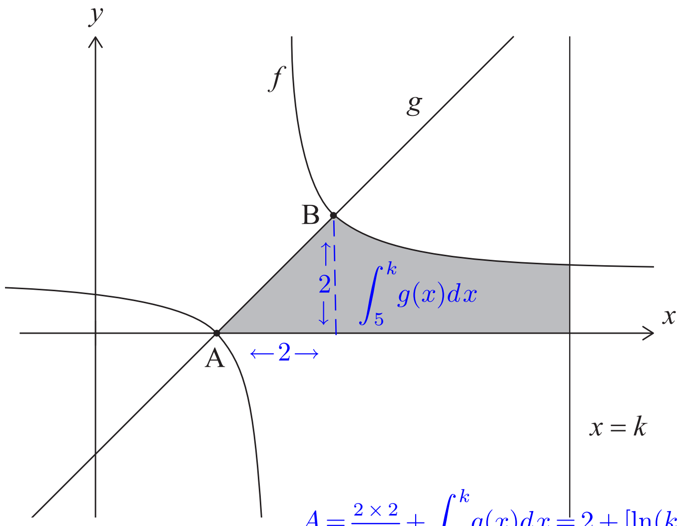
then $x_B = 5$

An other way, as we know that f and g intesect at $x_A = 3$, is to use the *symetry* then $x_B = 5$

The graphs of f and g intersect at points A and B. The coordinates of A are $(3, 0)$.

(a) Find the coordinates of B. $\boxed{B: (5, 2)}$ [5]

In the following diagram, the shaded region is enclosed by the graph of f, the graph of g, the x-axis, and the line $x = k$, where $k \in \mathbb{Z}$.



$A = \frac{2\times 2}{2} + \int_5^k g(x)dx = 2 + [\ln(k-4)+k] - [\ln(5-4)+5$

The area of the shaded region can be written as $\ln(p) + 8$, where $p \in \mathbb{Z}$.

(b) Find the value of k and the value of p. $2 + \ln(k-4) + k - 5 = \ln(p) + 8 \quad \Rightarrow \boxed{k = 11 \text{ and } p = 7}$ [10]

Problem 5 [/ 17 marks]

The function f is defined by $f(x) = 4^x$, where $x \in \mathbb{R}$

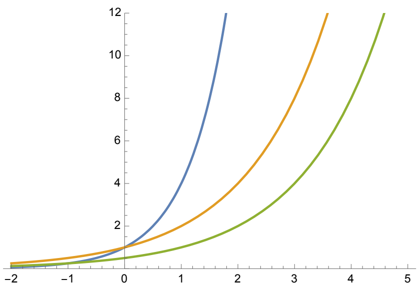
(a) $f^{-1}(x) = \log_4(x)$, hence $f^{-1}(8) = \log_4(3\times 4) = \log_4(2\times 4) = \log_4(2) + \log_4(4) = \frac{1}{2} + 1 = \boxed{\frac{3}{2}}$

(b) i) $g(x) = 1 + \log_2(x)$ then $g^{-1}(x) = 2^{x-1}$

ii) Comparing $g^{-1}(x) = 2^{x-1}$ and $f(x) = 4^x = 2^{2x}$ we notice that $f(x) = g(2(x+1))$

that correspond to : – a horizonal **contraction** of factor 2

– a horizonal *translation* to the left, of one unit.



(c) $(f\circ g)(x) = 4^{1+\log_2(x)} = 4\times 4^{\log_2(x)} = 4\times(2^2)^{\log_2(x)} = 4\times 2^{2\log_2(x)} = 4\times(2^{\log_2(x)})^2 = 4\times(x)^2 = 4x^2$

(d) The function h is denined by $h(x) = \frac{4x^2}{2x+1}$

(i) $2x - 1 + \frac{1}{2x+1} = \frac{(2x-1)(2x+1)+1}{2x+1} = \frac{4x^2}{2x+1}$

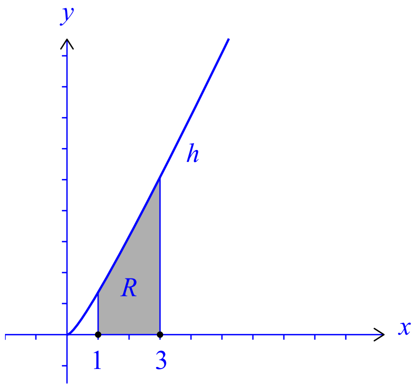
(ii) By (i): $h(x)=2x-1+\frac{1}{2x+1}$

therefore $\int h(x)dx=\int\left(2x-1+\frac{1}{2x+1}\right)dx=x^2-x+\frac{\ln(2x+1)}{2}+c$

$$\Rightarrow R=\int_1^3\left(2x-1+\frac{1}{2x+1}\right)dx=\left[x^2-x+\frac{\ln(2x+1)}{2}+c\right]_1^3$$

$$=\left(9-3+\frac{\ln(7)}{2}+c\right)-\left(1-1+\frac{\ln(1)}{2}+c\right)$$

$$=\boxed{6+\frac{\ln(7)}{2}\quad p=6,q=\frac{1}{2},r=7}$$



Problem 6 [/ 16 marks]

A particle P moves along the x-axis. The velocity of P is $v\,\mathrm{m\,s}^{-1}$ at time t seconds, where $v(t)=4+4t-3t^2$ for $0\le t\le 3$. When $t=0$, P is at the origin O.

(a) (i) Find the value of t when P reaches its maximum velocity.

(ii) Show that the distance of P from O at this time is $\frac{88}{27}$ metres. [7]

(b) Sketch a graph of v against t, clearly showing any points of intersection with the axes. [4]

(c) Find the total distance travelled by P. [5]

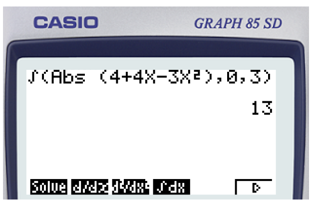
(a) (i) v is *maximum* $\Rightarrow a=\frac{dv}{dt}=0\Rightarrow -6t+4=0\Rightarrow \boxed{t=\frac{2}{3}s}$

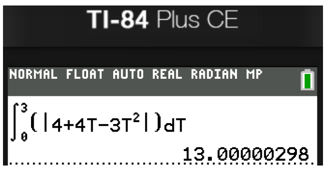
(ii) $d_{OP}=\int_0^{\frac{2}{3}}4+4t-3t^2dt=\boxed{\frac{88}{27}\cong 3.26m}$

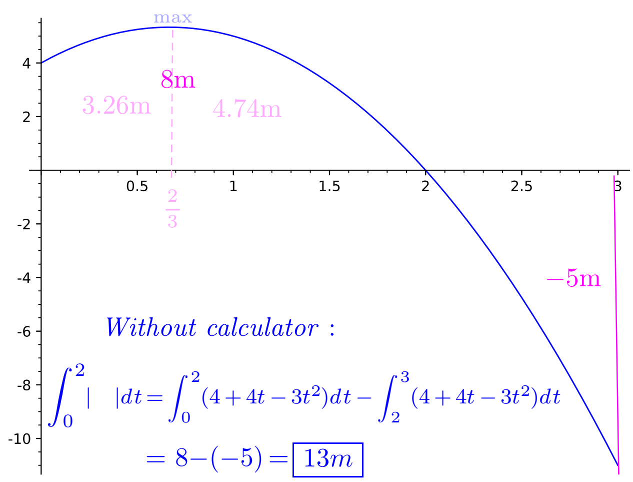
(b) There is an intersection at $t=2s$, hence $v(2)=0\,\mathrm{ms}^{-1}$

(c) The total distance traveled by P is $\int_0^2|4+4t-3t^2|dt$

with calculator







Without calculator :

$$\int_0^2|\quad|dt=\int_0^2(4+4t-3t^2)dt-\int_2^3(4+4t-3t^2)dt$$

$$=8-(-5)=\boxed{13m}$$

Problem 7

[/ 5 marks]

Box 1 contains 5 red balls and 2 white balls.
Box 2 contains 4 red balls and 3 white balls.

(a) A box is chosen at random and a ball is drawn. Find the probability that the ball is red. [3]

Let A be the event that "box 1 is chosen" and let R be the event that "a red ball is drawn".

(b) Determine whether events A and R are independent. [2]

(a) $\mathrm{P(R}=\frac{1}{2}\times\frac{5}{7}+\frac{1}{2}\times\frac{4}{7}=\boxed{\frac{9}{14}}$

(b) $\mathrm{P}(R|A)=\frac{P(R\cap A)}{P(A)}=\frac{\frac{1}{2}\times\frac{5}{7}}{\frac{1}{2}}=\frac{5}{7}$

$\Rightarrow$ $\boxed{A, R\ not\ \text{independant}}$

Problem 8

[/ 13 marks]

A discrete random variable, X, has the following probability distribution, where $a>0$ and k is a constant.

x	0	a	$2a$	$3a$
$\mathrm{P}(X=x)$	k	$3k^2$	$2k^2$	k^2

(a) Show that $k=\frac{1}{3}$. [5]

(b) Find $\mathrm{P}(X<3a)$. [2]

(c) Find $\mathrm{P}(X\geq a|X<3a)$. [3]

(d) Given that $\mathrm{E}(X)=20$, find the value of a. [3]

(a) $k+3k^2+2k^2+k^2=1 \quad \Rightarrow 6k^2+k-1=0 \qquad \Delta=25 \quad k=\frac{-1\pm 5}{12} \quad \text{as } k>0\text{: } k=\frac{4}{12}=\frac{1}{3}$

(b) $\mathrm{P(X<3a)}=k+5k^2=\boxed{\frac{8}{9}}$

(c) $\mathrm{P}(X\geqslant 2a|X<3a)=\frac{P(X\geqslant 2a\cap X<3a)}{P(X<3a)}=\frac{P(X=2a)}{P(X<3a)}=\frac{2k^2}{k+5k^2}=\frac{\frac{2}{9}}{\frac{8}{9}}=\frac{2}{8}=\boxed{\frac{1}{4}}$

(d) $\mathrm{E}(X)=k\times 0+3k^2a+2k^2 2a+k^2 3a\ =\ 10ak^2=20 \quad \text{with } k=\frac{1}{3} \quad \text{then} \quad \boxed{a=18}$